NOETHERIAN MODULE MATRIX OVER COMMUTATIVE RING

by

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Dedicated to

Beloved Mother and Grandmother

My Brother and Sister

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF NOTATIONS

$\oplus$	An addition operation that defined
$\otimes$	A multiplication operation that defined
$\otimes_n$	Multiplication modulo n
$\subseteq$	Subset
$\cong$	Isomorphism
$*$	Binary operation
$\bullet$	Scalar multiplication
$\mathbb{N}$	Natural number
R	Algebraic structure ring
M	Algebraic structure $R-$module
A_n	Square matrix size $n \times n$
$A_{m\times n}$	Matrix size $m \times n$
$M_n(R)$	Set of Square matrix size $n \times n$ with the entries are element of R
$M_{m\times n}(R)$	Set of matrix size $m \times n$ with the entries are element of R
$\blacksquare$	The end of definition, theorem, lemma, corollary, and proposition

Modul Noetherian Matriks atas Gelanggang Komutatif

ABSTRAK

Struktur algebra adalah merupakan set tak kosong yang disertai dengan operasi dedua yang memenuhi beberapa aksiom operasi dedua, seperti kumpulan, gelanggang, dan modul. Modul M dinyatakan modul Noetherian jika syarat ciri rantai menaik dipenuhi untuk setiap submodul bagi M. Andaikan R adalah gelanggang dan $M_{m\times n}(R)$ adalah set matriks dengan kemasukan bagi unsur dari gelanggang R, dan jika $m=n$ maka $M_n(R)$ disertai dengan penambahan dan pendaraban untuk matriks adalah suatu gelanggang dikenali sebagai gelanggang matriks. Tujuan utama kajian ini adalah untuk membina struktur algebra daripada set matriks untuk memenuhi setiap aksiom pada modul. Keperluan untuk gelanggang matriks $M_n(R)$ dan modul matriks $M_{m\times n}(R)$ memenuhi Noetherian akan diselidiki. Syarat Noetherian bagi struktur algebra akan dikaji, kemudian struktur algebra bagi kemasukan dalam matrik akan ditentukkan sehingga memenuhi modul Noetherian. Dalam kajian ini, ditunjukkan bahawa modul $M_{m\times n}(R)$ memenuhi $R-$modul Noetherian jika struktur algebra $(R,+)$ merupakan kumpulan Noetherian dan modul $M_{m\times n}(R)$ juga merupakan modul Artinian. Fakta berkaitan dengan homomorfisma modul untuk $M_{m\times n}(R)$ juga diperolehi.

Noetherian Module Matrix over Commutative Ring

ABSTRACT

Algebraic structure is a set together with one or more binary operations that satisfy some axioms of the binary operations, for example groups, rings, and modules. Modules M is Noetherian modules if the ascending chain condition hold for every submodules of M. Let R be a ring and $M_{m\times n}(R)$ is set of matrices which the entries are element of R, if $m=n$ then $M_n(R)$ together with addition and multiplication matrix is a ring called matrix ring. The aim of this study is to construct the algebraic structures from the set of matrices in order to fulfill every axioms of modules. The requirements for matrix ring $M_n(R)$ and module matrix $M_{m\times n}(R)$ that satisfy the Noetherian conditions are investigated. The Noetherian term of algebraic structures will be studied, then the algebraic structure for the entries of the matrix will be determined such that satisfy the Noetherian module. This study, it has been shown that modules $M_{m\times n}(R)$ is satisfy Noetherian $R-$module if the algebraic structure $(R,+)$ is Noetherian group and module $M_{m\times n}(R)$ is also Artinian module. The facts regarding homomorphism modules of $M_{m\times n}(R)$ are also obtained.

CHAPTER 1

INTRODUCTION

1.1 Introduction

A set is collection of well-defined objects (Jech, 2002). Set theory is the basic concepts on mathematics. Study about set theory influenced logic, because in order to solve and understanding some concept of set theory cannot be proven directly (Hintikka, 1998). Therefore, sometimes in lecture before we study about set theory lecturer will explain about mathematical logic. Algebra is one of the parts of mathematics that studies about sets and its properties. Abstract algebra is part of advanced topics in algebra that deal with algebraic structures that is the relation between binary operations and sets (Fraleigh, 2003). The basic algebraic structures are group, ring. Module theory learn about algebraic structures formed from two algebraic structures (group and ring). Before the discussion on module continues, the algebraic structures of group and ring will be introduced.

Algebraic structures refers to sets with one or more binary operations that satisfy some axioms. An operation is said to be binary on a set A if for every two elements of A, the result of the operation also element of A. The type of the algebraic structures depends on the axioms of the binary operations that fulfilled for the sets. The axioms of the binary operations on set are (i.) Commutative: for all $x, y \in A$ we have

$x \oplus y = y \oplus x$ and $x \otimes y = y \otimes x$;

(ii.) Associative: for all $x, y, z \in A$ we have

$(x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \otimes y) \otimes z = x \otimes (y \otimes z)$;

(iii.) Distributive: for all $x, y, z \in A$ we have

$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ and $(x \oplus y) \otimes z = (x \oplus z) \oplus (y \otimes z)$;

(iv.) There exists identity elements 0 and 1 such that for all $x \in A$ we have

$x \oplus 0 = x$ and $x \otimes 1 = x$;

(v.) For all $x \in A$ we have $-x, x^{-1} \in A$ such that

$x \oplus (-x) = 0$ and $x \otimes x^{-1} = 1$.

So, the type of an algebraic structure dependent on axioms of binary operation that fulfilled (Fraleigh, 2003).

The algebraic structures construct from a set and a binary operation consist of:

(i) **Semigroup**: A nonempty set together with a binary operations and associative for every element of the set.

(ii) **Monoid**: Semigroup with binary operation which contain an identity element.

(iii) **Group**: Monoid with binary operation which contain inverse element for every element on the set.

(iv) **Abelian group**: A group with the binary operation and commutative condition.

While, the algebraic structure formed from a set and two binary operations are:

(i) **Ring**: A nonempty set together with two binary operations, addition and multiplication, the addition operation is Abelian group and the multiplication operation is semigroup.

(ii) **Commutative ring**: A ring which the multiplication operation satisfy the commutative condition.

(iii) **Field**: Ring with the multiplication operation is an Abelian group.

So, properties of the binary operations is the basic key for determining an algebraic structure type, a little change in the binary operation axioms affect the types of the algebraic structure (Dummit & Foote, 2003).

In linear algebra, vector space, which is an algebraic structure built from an Abelian group and field as the corresponding scalars is studied (Anderson & Feil, 1950). Algebraic structure module is a generalization of the vector space where the scalar is a ring rather than field.

Module: An algebraic structure which built from two algebraic structures namely Abelian group and ring. Then, defined an operation between element of Abelian group and element of ring that satisfies

$$a(m+n) = am + an \ .$$

$$(a+b)m = am + bm \ .$$

$$(ab)m = a(bm) \ .$$

$$1_R m = m \ .$$

for every a,b element of ring and m,n element of Abelian group (Cameron, 2008).

1.2 Research Background

Matrix is rectangular array of number or expression arranged in rows and columns. Two matrices can be added and subtracted if they have the same sizes. Then, two matrices can be multiplied if the number of the columns in the first matrix equal to the number of the rows in the second matrix. The matrix learned in elementary linear algebra is matrix which the entries are element of real number and complex number (Anton & Rorres, 2005). Matrix over commutative ring is the matrix with the entries are element of commutative ring (Brown, 1993). Matrix over commutative ring is generalization of matrix where the entries are element of real number and complex number, because the entries of the matrix is ring rather than field in this case real number and complex number. The author also explained some basic properties of matrix over commutative ring and algebraic structure matrix ring. Matrix ring is an algebraic structure constructed from the set of matrix over ring together with addition and multiplication matrix.

Module theory is often assessed as a research topic on abstract algebra related to the properties and structures of the module itself. There are many kinds of module based on the sets or properties build a module such as

(i) **Torsion module:** if every elements of M are torsion element.

(ii) **Torsion-free module:** if the torsion element is zero (identity element of the module).

(iii) **Free module:** if it has a basis.

(iv) **Noetherian (respectively Artinian) module:** if every submodules of M fulfilled the ascending chain condition (respectively descending chain condition).

The types of this module was discussed in abstract algebra books (Adkin & Weintraub, 1992; Dummit & Foote, 2003; Goodman, 2006; Judson, 2009).

The term, “Noetherian” is named after Emmy Noether as the discoverer of that conditions. There are many studies about Noetherian module regarding to the properties of the Noetherian condition or the submodules of the Noetherian module until recent days. According to Brown (1993), besides describing on matrix ring, he also describe about finitely generated module (Brown, 1993). Noetherian module is also a finitely generated module. The author defined finitely generated module and some theorems related the finitely generated module. There are many studies about matrix on abstract algebra related to algebraic structures and the properties which formed by conducting a study on the entries of the matrix. This is because some properties, the result of operations, and definition on the matrix depends on the elements of the matrix entries. In 2005, there is a study about matrix module using set of square matrix as the Abelian group (Cao & Zhang, 2005). The aim of this research is to build an algebraic structure from the set of matrices size $m \times n$ which is used from matrix that the entries are element of commutative ring. This generalization is expected in a way that the algebraic structure can satisfies for every axioms of module. Furthermore, the set of matrices can fulfill the Noetherian module condition.

1.3 Problem Statement

Based on the background, the purpose of this research is to reconstruct the algebraic structures from the set of matrix in order to fulfill every axioms of the module. This can be implemented by generalized the set of square matrices to set of matrices size $m \times n$.

The algebraic structure is said to be module if the algebraic structure fulfill every axioms of right and left module (Adkin & Weintraub, 1992). Therefore, the algebraic structure which is appropriate for the entries of the matrix and the scalar ring in order to satisfy every axioms of right and left module need to be determined.

Noetherian module is module which every submodules of the module fulfilled the ascending chain condition (Goodearl & Warfield, 2004). It shows that the main focus in the study of Noetherian module is the submodules. Based on the background, it can be seen that the properties of matrix depend on the entries and the operations. Besides determining the algebraic structure for the entries of the matrix that can build module matrix, the algebraic structures satisfy the ascending chain condition term for every submodules of the module matrix also need to be considered.

1.4 Research Objectives

The objectives of this research are

(i) to investigate the Noetherian properties related group, ring, and module.

(ii) to determine the algebraic structures for entries of the matrix.

(iii) to validate that every axioms on module theory fulfilled for set of matrix.

(iv) to investigate and propose some conditions related Noetherian module for set of matrix.

1.5 Scope of the Research

This research focused on forming the algebraic structures based on matrix over commutative ring. The algebraic structure which built from set of matrix will be investigated. This research concerning with the formation of set of matrix which fulfilled the axioms of module theory based on the elements of the matrix. And also, some algebraic structures of set of matrix that will support the forming of the module structure especially Noetherian module will be described.

1.6 Significant of the Research

Based on the problem statements and objectives of this research, the construction of the algebraic structure module and the Abelian group formed by set of matrix will be investigated. In this study, matrix is the main core in forming the algebraic structure and the entries of the matrix is the important factor to solve some problem occurred. Noetherian module and the properties may be obtained for the matrix case.

Study about module theory and matrix theory have been done in tandem until recent days. Additionally, study about algebraic structure of matrix ring, set of matrix as the set together with addition and multiplication matrix, also have become the main interest by many researcher until recent days. However, study about algebraic structure module with matrix as the main set is difficult to find and have not been found in some international journals. Study about module theory especially Noetherian module was also rarely been done in the last 10 years. Naghipour (2005) used Noetherian module to support his study about Ratliff-Rush closures. Afterward, in 2012 and 2015 there were studies about some properties of Noetherian module (Aldosary & Alfadli, 2012; Cuong, Quy, & Truong, 2015). This shows that study about Noetherian module becomes less interested in. In addition, study about algebraic structure matrix ring has been continously assessed but study about module matrix cannot be found.

The findings of this study will redound to the development of research on abstract algebra especially module matrix, considering that study of module matrix have started to be learned in 2005 by Cao & Zhang. Furthermore, the study increase references about Noetherian module which is less attractive lately.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The literature review is an important aspect in a study, it enable researchers to achieve a better understanding of the problem in a study. The literature review discussed about the current issue which still becoming a discussion by researchers until nowadays. There are two significant matters will be discussed in this chapter including some study related to matrix theory and algebraic structures related matrix and module. Moreover, some definitions that will be used in solving some problems in this study will be given.

2.2 Matrix Theory

Matrix is a rectangular array of number or expression arranged in rows and columns (Anton & Rorres, 2005). Matrix is part of linear algebra. In linear algebra, matrix is used to resolve some problems related to linear algebra itself, for example system of linear equation and graph theory. Matrix also used to solve some problems from other field study such as operation research. There are many study about matrix regarding the matrix theory itself or algebraic structures, such as study about properties that formed by matrix. According to Brown (1993), matrices over commutative ring is a generalization of matrix by changing the entries of the matrix with element of commutative ring. In the book also showed that the matrix operations for matrices over commutative ring is the same as the ordinary matrix operations in general. It also discussed the cases of matrices over commutative ring related to abstract algebra and linear algebra (Brown, 1993).

Recently, there are many studies on properties of the matrix concerning the algebraic structures. In Pazzis & Šemrl (2015) research, showed that some theories and condition of rectangular matrix on geometry where the entries are element of EAS division ring (Pazzis & Šemrl, 2015). The entries of the matrix is the important factor to help us in resolving some problems. Recently, study about matrix over galois ring has been done related to linear algebra (Guo & Li, 2015).

Besides matrix over commutative ring, matrix where the entries are element of polynomial ring also been assessed. Lissner (1961) study about square matrix over polynomial ring and some conditions of it algebraic structure (Lissner, 1961). Lately, study on matrix over polynomial ring also been done regarding to the properties of the matrix, for example study of ideals from set of matrix which contains polynomial on it (Rissner, 2016).

2.3 Algebraic Structure of Matrix Ring

Set of matrix with the entries are element of ring associated with addition and multiplication matrix is called matrix ring (Hartley & Hawkes, 1970). Matrix ring is a specific example of algebraic structure ring, the other example is polynomial ring. Matrix ring has a unique case where the algebraic structure cannot fulfill the commutative axiom of multiplication operation for every element of the ring. This condition make many researchers interested to study about inverse element of the matrix (Bu, Zhao, & Zeng, 2008; Bu, Zhao, & Zhang, 2009; Cao & Li, 2009; Sheng, Ge, & Cao, 2013; Zhang & Bu, 2012). There are many studies about matrix ring that have been done related the matrix ring as the main object of the study or to support study about another algebraic structures. Matrix ring and some properties of the algebraic structure can be learned more by changing the entries of the matrix with specific algebraic structure (Lissner, 1961; Berman, 1975). Furthermore, study about structure of the matrix ring also can be done by taking some conditions on its matrix itself (Wyk, 1996). This shows that many results can be obtained just by studying on the matrix ring itself.

Matrix ring can also be used to support study about another algebraic structures. Study about relations of two algebraic structures and its properties indeed interesting to study. A few studies have been done for some conditions of the matrix such as the type of the matrix or the properties of the matrix related to the algebraic structure of the matrix (Dascalescu et al., 1999; Haghany & Varadarajan, 2000; Busque & Simon, 2003). Furthermore, there are many studies about matrix ring. For example Radjabalipour, Rosenthal, & Yahaghi (2004), study about Burnside theorem for matrix ring case by changing the entries of the matrix with element of division ring. Study about matrix ring by changing the entries of the matrix performed to investigate the properties of the algebraic structures of matrix ring for example strongly clean condition for matrix ring (Borooah, Diesl, & Dorsey, 2008; Chen, Yang, & Zou, 2006; Li, 2007).

In abstract algebra, especially algebraic structures also study about the relations between two algebraic structures called homomorphism, for example group homomorphism, ring homomorphism, and module homomorphism. Homomorphism also play an important role to solve some cases in matrix ring. There are many researchers used homomorphism as the object of study or to assist the matrix ring study in order to resolve some problems emerged (Anh & Wyk, 2011; Breaz et al., 2013; Kuzucouglu, 2001).

2.4 Algebraic Structure of Noetherian Module

Algebraic structure is the sets together with binary operations that satisfy some axioms (Fraleigh, 2003). There are many kinds of algebraic structures for example group, ring, field, vector space, module, etc. Group is an algebraic structure built from a set and a binary operation that satisfy some axioms. Ring is an algebraic structure built from a set and two binary operations that satisfy some axioms. While, module is an Abelian group together with scalar multiplication of scalar ring that satisfy some axioms. Module are generalization of the vector space which the scalars multiplication element of ring (Adkin & Weintraub, 1992). There are many studies has been done related to the type or characteristic of the module. There are many studies has been done related module theory, for example study about relation between algebraic structures especially polynomial ring and module theory. Endo (1963), study an algebraic structure of projective module on polynomial ring. The study investigated all relevant to the polynomial ring by changing the coefficient of the polynomial. Many conditions can be obtained from those changes (Endo, 1963). The study on projective module over polynomial ring started from Serre Conjecture (Serre, 1955). The study about projective module over polynomial ring continued with the ring on polynomial more specifically (Kang, 1979). Module theory also learned through linear algebra by using the polynomial matrix to obtain an algebraic structure module (Lin, 1999).

Noetherian condition is a specific condition that formed on the algebraic structures. Noetherian is a finiteness condition on a set which is every subsets of an algebraic structure satisfy the ascending chain condition (Grillet, 2007). Noetherian condition can also be applied to algebraic structure group and ring, provided that every structures has their own rules for the ascending chain condition. Furthermore, algebraic structures that satisfy ascending chain condition is called Noetherian group, Noetherian ring, and Noetherian module (Adkin & Weintraub, 1992; Wisbauer, 1991; Zassenhaus, 1969). Matlis (1958), study about injective module that formed from noetherian ring. Not only noetherian ring, the author also investigate the homomorphism relationship of commutative ring, complete ring, and local ring (Matlis, 1958). Furthermore, study about Noetherian conditions of ring and module draw attention many researchers. Study about the Noetherian module and Noetherian injective ring regarding to uniform condition on module and ring also have been investigated (Feller, 1964).

Formanek (1973) described another way to proof Eakin-Nagata Theorem by using the Noetherian properties. Recently, there are many studies have been done about Noetherian module related to the properties of Noetherian module. For example, study about Noetherian module regarding the length of the Noetherian module and also krull ordinal of the submodules (Rhodes, 1974). The study of krull ordinal on Noetherian module also produces a cancellation properties by using Krull-Schmidt Theorem (Brookfield, 2002). In addition, Noetherian module also can be used to assist in proving some theorem of another study of abstract algebra (Tyc & Wisniewski, 2003). Lately

Cuong, Quy, & Truong (2015), investigate about the number of irreducible submodules of Noetherian module.

Next, the locally Noetherian module will be discussed. A Module is said to be locally Noetherian module if every submodules satisfy Noetherian module (Alizade & Buyukasik, 2008). Research by Arnold & Brewer (1971), showed about locally Noetherian ring which also Noetherian ring for commutative ring. The study explain that a ring R is said to be locally Noetherian, if for maximal ideal I of R then R_I is Noetherian (Arnold & Brewer, 1971). Study about locally Noetherian ring for commutative ring continued for prime ideals of ring (Heinzer & Ohm, 1971). Jabbar (2015), investigate locally Noetherian ring regarding to Jacobson radical of the ring. Locally Noetherian module has a similar concepts with locally Noetherian ring. Locally Noetherian ring depend on the ideals and locally Noetherian module depend on the submodules. Zhong-kui & Fang (2000), studied some characteristic of locally Noetherian module related to the injective module.

2.5 Theoretical Background

2.5.1 Binary Operation

Before discussing about algebraic structures, the definition about binary operation and its properties will be given.

Definition 2.1 (Fraleigh, 2003): A binary operation $*$ on set S is a function mapping $S \times S$ into S. For each $(a,b) \in S$, we will denote the element $*((a,b))$ of S by $a * b$.