A STUDY ON THE EXISTENCE AND UNIQUENESS OF A CLASS OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

by

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LIST OF ABBREVIATIONS

FDEs	Fractional Differential Equations
FDO	Fractional Differential Operator
ODEs	Ordinary Differential Equations
m-$P\Lambda\Lambda$	*m*-pseudo almost automorphic
$P\Lambda\Lambda$	pseudo almost automorphic
FNDEs	Fractional Neutral Differential Equations
NDEs	Neutral Differential Equations
NDDE	Neutral Delay Differential Equations
NDD	Neutral Delay Differential
DDE	Delay Differential Equations

LIST OF SYMBOLS

D^{μ}	Riemann–Liouville fractional differential operator (first order).
$D^{2\mu}$	Riemann–Liouville fractional differential operator (secondorder).
$D^{3\mu}$	Riemann– Liouville fractional differential operator (third order).
I^{μ}	Riemann–Liouville Fractional Integral
Γ	Gamma function
max	Largest value of the function
L_{∞}	The space of essentially bounded functions
E_q	The Mittag-Leffler function in one parameter
sup	The least upper bound
$\Re$	The set of real numbers
$\Re^n$	*n*-dimensional space
X and Y	Be two Banach spaces

KAJIAN MENGENAI KEWUJUDAN DAN KEUNIKAN SATU KELAS PERSAMAAN PEMBEZAAN PECAHAN

ABSTRAK

Kalkulus pecahan merupakan satu cabang yang penting dalam bidang analisis matematik. Ianya telah dipersetujui dengan permintaan dan eksplorasi terbitan dan kamiran mengikut peringkat rawak. Pergerakan berkala adalah fenomena yang sangat penting dan istimewa bukan sahaja dalam sains semula jadi tetapi juga dalam sains sosial. Secara umumnya, bukan semua persamaan pembezaan mempunyai penyelesaian berkala. Namun, terdapat kes istimewa persamaan pembezaan yang menarik perhatian untuk mendapatkan penyelesaian berkala. Persamaan pembezaan tersebut adalah persamaan pembezaan neutral dan persamaan pembezaan Rayleigh. Persamaan pembezaan Rayleigh mempunyai banyak aplikasi dalam fizik dan biologi. Ia digunakan dengan keputusan eksperimental yang berkala. Kewujudan hasil berkala dan positif telah diwujudkan dalam kaedah baru. Beberapa hasil kerja dalam literasi telah dibangunkan dan diperluaskan pada kewujudan dan keunikan kelas-kelas persamaan pembezaan pecahan, iaitu persamaan pembezaan pecahan, persamaan pembezaan pecahan Rayleigh, persamaan pembezaan pecahan neutral Rayleigh. Semua kelas persamaan pembezaan pecahan ditafsirkan mengikut kalkulus pecahan Riemann-Liouville. Tesis ini memfokuskan kepada perkembangan kalkulus pecahan untuk memperluaskan dan mengitlakkan banyak kelas persamaan pembezaan pecahan. Skop utama kajian ini adalah persamaan pembezaan neutral pecahan, persamaan pembezaan Rayleigh pecahan dan persamaan pembezaan neutral Rayleigh pecahan. Kajian ini telah menerbitkan keputusan baru pada kewujudan dan keunikan persamaan pembezaan neutral pecahan, persamaan pembezaan neutral Rayleigh pecahan dan persamaan pembezaan neutral Rayleigh pecahan. Satu model matematik persamaan pembezaan pecahan telah dikaji dengan pengitlakan persamaan neutral peringkat pertama. Komposisi teorem untuk fungsi m-PΛΛ dengan syarat yang bersesuaian telah dibuktikan berdasarkan teori interpolasi dan teorem titik tetap Banach.

Oleh itu, penyelesaian untuk kes ni adalah unik. Oleh kerana kalkulus pecahan Riemann-Lioville adalah bukan berkala, maka satu penciptaan untuk mendapatkan keberkalaan kelas-kelas persamaan pembezaan pecahan telah diperkenalkan. Satu persamaan jenis Rayleigh dengan kelewatan yang bergantung kepada keadaan telah dipertimbangkan dalam tesis ini. Kewujudan penyelesaian berkala kepada persamaan ini telah diselidiki. Semasa pengitlakan, terbitan pecahan Riemann-Liouville telah digunakan dan syarat cukup untuk kewujudan penyelesaian berkala diperolehi.

A Study On The Existence And Uniqueness Of A Class Of Nonlinear Fractional Differential Equations

ABSTRACT

Fractional calculus is the important branch of mathematical analysis field. It covenants with the requests and exploration of derivatives and integrals of random order. Periodic motion and special phenomena are very important not only in natural science, but also in social science. In general, not all differential equations have periodic solutions. But there exist special differential equations which are interesting to get their periodic solutions, which are neutral and Rayleigh differential equations. Rayleigh differential equation has many applications in physics and biology. It deals with experimental results with periodicity. The existence of periodic and positive outcome is established in a new method. Some available work in the literature are developed and extended on the existence and uniqueness of a class of fractional differential equation. All the classes of fractional differential equations are interpreted under Riemann-Liouville fractional calculus. This thesis focuses on the development of fractional calculus to extend and generalize many classes of fractional differential equations. The main scope of this study is fractional neutral differential equation, fractional Rayleigh differential equation and fractional neutral Rayleigh differential equation. This study establishes new results on the existence and uniqueness of fractional neutral, fractional Rayleigh and fractional neutral Rayleigh differential equations. A mathematical model was studied by generalization of the neutral differential equation of the first order. A composition theorem for m-PΛΛ functions under appropriate conditions was proved based on interpolation theory and Banach's fixed point theorem. Therefore, the solution, in this case, is unique. As the Riemann–Liouville fractional calculus was not periodic, a construction to get the periodicity of some classes of fractional differential equations was introduced. A Rayleigh-type equation with state-dependent delay was considered and the existence of periodic solutions to this equation was investigated. During the generalization, Riemann–Liouville fractional derivatives was utilized and sufficient conditions for the existence of periodic solutions were obtained.

CHAPTER 1: INTRODUCTION

1.1 General Background

Fractional calculus is a branch of mathematics that evaluates multi-order integrals and derivatives. It has been intensively studied for the past 30 years due to its usefulness in science and engineering. Fractional calculus goes back to Euler however, its extensive usage is a recent phenomenon (Own & Me, 2012). Currently, researchers who work on dynamic systems extensively utilize fractional differential equations (FracDE). FracDEs are more representative of real world problems relative to its integer counterpart (Mistry, Khan, & Suthar, 2016).

FracDEs are extensively used around the globe for various purposes (Xiaojun, 2012), due to their generalization of ordinary differential equations (ODEs) for a random (non-integer) order. There are more than a dozen titles on this subject in literature (Gorenflo & Mainardi, 2008), such as the encyclopedic treatise by Samko, (1995) being the most cited. Also, the treatise by Gorenflo and Vessella (1991) is made up of intricate analysis of mathematics and/or real-world applications of fractional calculus, although this was not made apparent by its subject matter. Recent renewed interest in fractional calculus is attributed to its ubiquity in numerical analysis, physics and engineering, and also fractal phenomena.

The analysis of fluid mechanics, mathematical biology, viscoelasticity, rheology, electrical networks, chemical physics via fractional derivatives result in expressions termed (FracDEs) (Bonilla, Rivero, Rodríguez-Germá, & Trujillo, 2007; Podlubny & Thimann, 1998). Agarwal; Al-Mdallal; Cho; and Jain, (2018), was the first to write extensively on FracDEs, where they rehashed the theory of FracDEs and its corresponding applications. Recently, Podlubny and Thimann, (1998) and Alzabut, Abdeljawad, and Baleanu, (2018) devoted their efforts fully on the theory and potential applications of FracDEs. These authors used many fractional calculus approaches to solve FracDEs, such as the fractional Green's function method, the Laplace transform method, the Mellin transform method, the power series and Babenko's symbolic calculus method, method of orthogonal polynomials, and other numerical methods. However, most are applicable to specific cases of FracDEs, and whether or not they are generalizable remains unclear.

Periodic motion is vital in science, engineering, and social science via climate, food supplement, insecticide population, and sustainable development. Periodic solutions are indeed expected in differential equations, making up one of the most important research topic in differential equations. They are also expected to be present in dynamical systems, and constitute the bulk of research effort in the theory of dynamical systems, encompassing applications such as celestial mechanics, biology, and finance. (FracDEs) represent the most vital approach to generalization in ODEs (Lizama & Poblete, 2011). Recent developments in physics, engineering, biological sciences, and other fields confirmed that system dynamics can be described by FracDEs, and delayed FracDE is more representative in its description of natural phenomena (Budhia, Kir, Gopal, & Kiziltun, 2016).

Lord Rayleigh, (1896) (Physics Nobel Laureate, 1904) came up with an equation:

$$x''(t) + f(x'(t)) + ax(t) = 0 \tag{1.1}$$

which can be used to model the oscillations of a clarinet reed (acoustics) Wang and Dai,(2009).

The periodic solutions for Rayleigh equations containing two deviating arguments were analyzed by Sirma, Tunc, and Özlem, (2010). The periodic solutions are expected and extensively studied. They are also expected to be present in dynamic systems, and are extensively studied in that context as well. FracDEs are especially effective in modelling complex phenomena (Lizama & Poblete, 2011).

The periodicity is the most important part in time-delay systems solutions. (Peng, Liu, Zhou, & Huang, 2006; Huang, He, Huang, & Tan, 2007; Sirma et al., 2010) pointed out that periodic solutions are vital in differential equations and dynamical systems, and this study is an attempt to elucidate this matter further. Rayleigh differential equation is ubiquitous and is present in mechanics, physics, and engineering.

The oscillatory properties of solutions of second-order neutral differential equations have garnered the attention of theorist and practitioners in this field. Literature is filled with examples of oscillation and asymptotic behaviour of neutral equations with discrete deviating arguments. Recent research efforts have been focussed on obtaining suitable conditions for oscillatory and non-oscillatory behaviours of multiple classes of neutral differential equations with distributed deviating arguments (Zhang, Baculíková, Džurina, & Li, 2016).

The mathematics involved in elucidating fractional calculus was only defined at the end of the previous century. There was extensive modification required to accommodate many physical facts. The Riemann–Liouville fractional derivative utilizes integer-order initial conditions to define the fractional order differential equations of Podlubny and Thimann, (1998). Also, Kolwankar and Gangal, (1996) used the Riemann–Liouville fractional derivative to study the nowhere differentiable fractal functions reported by Loverro, (2004) Thereafter, Riemann–Liouville fractional differential operator (FracDO), which is a direct generalization of Cauchy's formula for the n^{th} iterated integral, was reported by Riemann, and it makes up an important part of fractional calculus.

1.2 Problem Statement

FracDEs can realistically model many science and engineering phenomena. Thus, solving FracDEs has been of intense research interest. Several methods have been proposed to do so, such as the Laplace transform (Miller & Ross, 1993; Podlubny & Thimann, 1998), the Mellin transform (Podlubny & Thimann, 1998), the iteration method (Samko, 1995), the Fourier transform (Kempfle, Schäfer, & Beyer, 2002) and the operational method. These aforementioned methods are applicable for specific FracDEs (Luchko & Srivastava, 1995; Luchko & Gorenflo, 1998), but it should also be pointed out that the majority of FracDEs cannot be generalized especially the non-linear ones, (Luchko & Gorenflo, 1998).

FracDEs are generalizations of the class of ordinary differential equations (ODEs) for a random (non-integer) order. FracDEs have expanded desirability and substantial interests due to their ability to simulate a complex phenomenon. These equations capture nonlocal relations in space and time with power-law memory kernels

(Liu et al., 2010). Not all equations have a periodic solution. Moreover, not all equations have interesting to get their periodic solutions. But there exist special cases of equations which are interesting to get their periodic solutions such as neutral and Rayleigh differential equations. Rayleigh differential equation has many applications in physics, biology and their topics deal with experimental results with periodicity.

The existence of periodic and positive outcome is established in a new method. Some recent works are develop and extend. On the existence and uniqueness of a class of fractional differential equation, namely fractional neutral differential equation, fractional Rayleigh differential equation, fractional neutral Rayleigh differential equation. All the class of fractional differential equations are interpreted under Riemann-Liouville calculus. The aim of this research to study the periodicity computation of this model by using neutral Rayleigh differential equations, which are recognized in various studies including biology.

1.3 Research Objectives

The aim of this study is to establish the existence and uniqueness of a class of fractional differential equations by achieving the following objectives:

(i) To obtain new results on the existence and uniqueness of fractional neutral differential equations.

(ii) To obtain new results on the existence and uniqueness of fractional Rayleigh differential equations.

(iii) To obtain new results on the existence and uniqueness of fractional neutral Rayleigh differential equations.

(iv) To obtain new mathematical models in Biology and Psychotherapy problems.

1.4 Research Questions

The research question of this work are:

(i) Could the neutral differential equation of the first order be generalized fractional order and obtain new results?

(ii) Could the Rayleigh differential equation of the second order be generalized the fractional order and obtain new results?

(iii) Could the neutral Rayleigh differential equation of the third-order be generalized the fractional order and obtain new results?

(iv) Could we obtain new mathematical models in Biology and Psychotherapy problems?

1.5 Research Scope and Limitations

This work involves developing fractional calculus to extend and generalize multiple classes of fractional differential equations. The main scope of this work includes fractional neutral differential equation, fractional Rayleigh differential equation, and fractional neutral Rayleigh differential equation.

1.6 Research Significance

This study is based on theoretical fractional equations, and divided into three parts to demonstrate its significance. The first part focuses on fractional neutral differential equation, the second focuses on fractional Rayleigh differential equation, and the third focuses on fractional neutral Rayleigh differential equation. This work defines new classes of neutral and Rayleigh differential equations using the existence and uniqueness periodic solutions, making it applicable in disciplines such as physics, engineering, and biology.

1.7 Research Flowchart

The study emphasizes the development of fractional calculus for the generalization of multiple classes of fractional differential equations. Some of the main equations include fractional neutral differential equation, fractional Rayleigh differential equation, and fractional neutral Rayleigh differential equation. The steps below will detail the general methodology of this work, as per Figure 1.1.

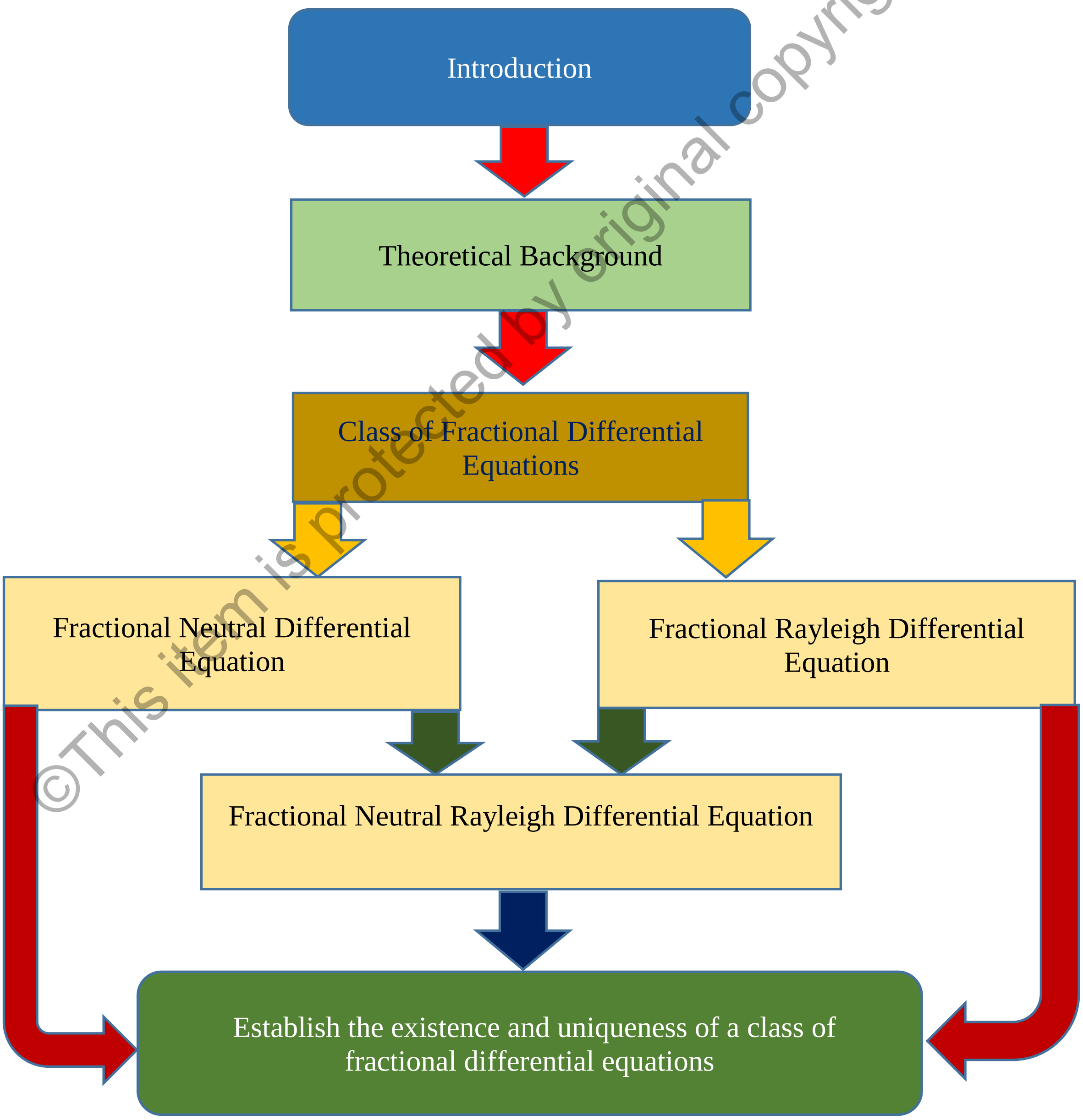

Figure 1.1 Flowchart of the study

1.8 Thesis Organization

This thesis is made up of six distinct chapters.

Chapter 1: Introduces fractional calculus, fractional differential equations, neutral and Rayleigh differential equations, and Riemann–Liouville fractional derivatives. The problem statement, research objectives, study questions, scope of study and the significance of this study will also be detailed.

Chapter 2: Provides a brief literature review describing periodic solutions, neutral differential equations, and Rayleigh differential equations.

Chapter 3: Presents the result on the existence and uniqueness first order of fractional neutral differential equations.

Chapter 4: Presents the result on the existence and uniqueness second order of fractional Rayleigh differential equations.

Chapter 5: Presents the result on the existence and uniqueness third order of fractional neutral Rayleigh differential equations.

Chapter 6: Conclude this work and recommend future possible works.

1.9 Summary

FracDE and analyzed problems reported by previous works have discussed. The objectives and contributions of the study were also briefly detailed. The method of the existence and uniqueness FracDE was also identified.

CHAPTER 2 : LITERATURE REVIEW

2.1 Introduction

This chapter contains a broad review of past investigations on the existence and uniqueness of fractional differential equations. This chapter is organised into four sections. Section 1 presents the fractional calculus with periodic solutions. Section 2 summarizes the neutral differential equations. Section 3 describes a Rayleigh differential equation while, Section 4 discusses the neutral Rayleigh differential equation and concludes the experimental results of the current study.

2.2 Fractional calculus

During the seventieth century, Newton and Leibniz derived the fractional calculus. The fractional differential equation is an abstraction of conventional differential equations and incorporation into arbitrary non-integer orders. It is a well-established method for characterizing various conditions in economy, engineering, physics, and science. Current studies have demonstrated fractional derivative formulation that enables precise characterization of numerous physical systems (Mistry, Khan, & Suthar, 2016) .

Due to its wider application, (FracDEs) are extensively used in the area of electrical circuits electroanalytical chemistry feedback amplifiers fractional multipoles viscoelasticity, and neuron modelling including diverse fields of physics chemistry as well as biological sciences (Mistry et al., 2016). Furthermore, numerous books and monographs have been published in this area. In addition, several studies demonstrated current advances in fractional differential and fractional integral differential equations with utilizations encompassing numerous promising operators of fractional calculus. In this regard, fractional order behaviour that could differ with time or space have been observed in several physical processes. Fundamentally, the fractional calculus has enabled the actions of incorporation and variation to any fractional order. The order could acquire any actual or theoretical value (Mistry et all., 2016). Lately, the theory of fractional differential equations has been widely addressed, in which the studies reported findings by applying fixed point theorems.

The fractional calculus is a field of mathematics study that grows out of the traditional definitions of calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. Although theory of fractional calculus (fractional derivatives and fractional integral) is well-established since twenty years ago, it might be expressed as an innovative subject. The paper that presented on the historical progress on the fractional calculus AL Horani ,and Khalil,(2018) is useful as a reference pertaining to this subject. The derivatives of fractional order include the complete history of the function in weighted forms, where the derivatives of integer order rely upon the local behavior of the functions.

The Riemann–Liouville differential operators of fractional order $0 < q < 1$, which is the part of differential equations, is anticipated to be essential in modelling numerous physical phenomena (Diethelm & Ford, 2004; Kiryakova, 2010; Samko, 1995). Hence, there should be an independent investigation on the theory of differential equations comparable to the important concept of ordinary differential equations. The current study emphasizes the established evidences of differential equations rather than inferring the fundamental feature and uniqueness of the fixed point concept (Lakshmikantham, Leela, & Rao, 1987). This approach allows identification of the dissimilarities and complexity of the analysis. The analysis commences with the elemental concept of inequalities that allows required evaluation findings that are valuable for investigating the qualitative and quantitative attributes of solutions of (FracDEs).

In recent years there has been progressive advancement in fractional calculus, particularly in terms of its practical utilization. Many utilization of fractional calculus are available in fluid dynamics, image processing stochastic dynamical systems plasma physics, nonlinear control concept nonlinear biological systems and quantum mechanics, that provide comprehensive information on the history and utilization of fractional calculus (Alzabut et al., 2018). Moreover, fractional derivatives offer precise models of actual issues compared to integer order derivatives. Furthermore, fractional derivatives also provide good means of description for existence and uniqueness of many resources as well as procedures.

The solvability of various categories of (FracDEs) is the key benefit provided by fractional derivatives compared to the classical integer order models (Budhia, Kir, Gopal, & Kiziltun, 2016). Furthermore, a previous study demonstrated the existence and uniqueness of solution for (FracDEs) encompassing Riemann-Liouville differential operators with integral boundary conditions by using the monotone iterative technique (Wang & Xie, 2008). On the other hand, another study reported the preliminary value issue for a category of fractional neutral functional differential equations and acquired the existence criteria relating to Krasnoselskii's fixed point theorem (Agarwal, Zhou, &

He, 2010). Furthermore, Momani and Odibat, (2007) presented comparisons between the solutions of the fractional order differential equations via homotopy perturbation and variational iteration techniques. Moreover, Ahmad, Ntouyas, and Tariboon, (2016) proposed a novel theory on the coupling of nonlocal integral circumstances, which evidenced the existence and uniqueness of solutions for a coupled system of fractional differential equations.

2.3 Neutral Differential Equations

The neutral differential equations are the mathematical models that are widely applied in electrical networks encompassing biology, ecology, electrical engineering, lossless transmission, mechanics, medicine. Fundamentally, neutral delay differential equations are known as (NDDEs), where the main order derivative belongs to the purpose available for both inclusive and devoid of derivatives (Das, Pandey, & Sukavanam, 2016).

A previous study presented the oscillatory behavior of solutions of (NDDE) that is composed as Eq (2.1) (Grammatikopoulos, Grove, & Ladas, 1986).

$$\frac{d}{dt}\left[y(t)+\beta p y(t-\tau)\right]+Q(t)y(t-\sigma)=0, \qquad t \geq t_0 \ . \tag{2.1}$$

In addition, the study also described the asymptotic behavior pertaining to non-oscillatory solutions, in which τ and σ are positive constants, $Q \in C([t_0,\infty),\Re^+)$ as well as the real parameter p exists within $\left[-1,0\right]$. Furthermore, the study demonstrated the behavior pertaining to the solutions of Eq. (2.1) owing to the entire potential results for p, specifically, in consideration of $p > -1$ and for $p > 0$. Moreover, the study also