TRIGONOMETRIC B-SPLINE BASED APPROACH FOR SOLVING INITIAL AND BOUNDARY VALUE PROBLEMS OF DISPERSIVE EQUATIONS

by

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LIST OF ABBREVIATIONS

PDE	Partial Differential Equation
BBMB	Benjamin-Bona-Mahony-Burger
MRLW	Modified Regularized Long Wave
MEW	Modified Equal Width
EW	Equal Width
1D	One –Dimensional
GEW	Generalized Equal Width
CAGD	Computer Aided Geometric Design
RMS	Root Mean Square
VI	Variational Iteration
HP	Homotopy Pertubation
VIM	Variational Iteration Method
HPM	Homotopy Pertubation Method
CuTBS	Cubic Trigonometric B-spline
CuHBS	Cubic Hybrid B-spline
QuTBS	Quintic Trigonometric B-spline
QuHBS	Quintic Hybrid B-spline
QuBS	Quintic B-spline
SeBS	Septic B-spline

LIST OF SYMBOLS

x_i	Grid point in x-direction
h	Step size in x -direction
Δt	Time step
T_i^k	i -th Trigonometric B-spline basis function of order k
B_i^k	i-th B-spline basis function of order k
H_i^k	i-th Hybrid B-spline basis function of order k
w	Dependent variable
x	Independent variable
t	Independent variable
a	Starting value of x
b	Ending value of x
M	Number of interval in t-dimension
N	Number of interval in x-dimension
k	Index for time
i	Index for space
f	Given function
x_i	Grid point in x-direction
h	Step size in x -direction

Pendekatan Berasaskan Trigonometri Splin-B Untuk Menyelesaikan Masalah – Masalah Nilai Mula Dan Sempadan Bagi Persamaan –Persamaan Berserak

ABSTRAK

Pelbagai kaedah berangka telah dibangunkan dengan menggunakan fungsi splin untuk menyelesaikan persamaan pembezaan separa (PPS) serakan seperti kaedah pembezaan terhingga (KPT), kaedah unsur terhingga (KUT) dan kaedah isipadu terhingga (KIT). Setiap kaedah mewarisi kelemahan tertentu seperti kerumitan, kos pengiraan yang tinggi dan memerlukan fungsi cubaan atau batasan untuk kes-kes tertentu. Berdasarkan kepada kekangan-kekangan tersebut, KPT digabungkan bersama fungsi splin-B telah diperkenalkan untuk menyelesaikan PPS. Matlamat utama tesis ini adalah untuk menerokai penyelesaian kepada PPS serakan yang tepat dan boleh dipercayai. Kaedah kubik trigonometri splin-B (KuTSB), kaedah kubik hibrid splin-B (KuHSB) dan dua kaedah baru yang dinamakan kaedah kuintik trigonometri splin-B (KuiTSB) dan kaedah kuintik hibrid splin-B (KuiHSB) telah dipilih bersama skim pembezaan terhingga untuk menyelesaikan PPS serakan peringkat ketiga yang digelar persamaan *Modified Regularised Long Wave (MRLW)*, persamaan *Benjamin-Bona-Mahony-Burgers (BBM-Burgers)* dan persamaan *Modified Equal Width (MEW)*. Kaedah-kaedah yang dicadangkan menghasilkan penyelesaian-penyelesaian berangka yang didapati lebih baik atau mematuhi dengan baik kaedah-kaedah daripada literatur. Perbandingan ralat maksima (L_∞) dan ralat *Euclidean* (L_2) daripada literatur juga dilakukan pada setiap contoh. Prestasi kaedah yang dicadangkan adalah dikenalpasti untuk menjadi lebih tepat dari kaedah KuTSB dan KuiTSB. Dalam menganalisis kestabilan kaedah yang dicadangkan, *Von Neumann* analisis kestabilan telah digunakan kepada skim yang telah dilinearkan. Skim-skim tersebut telah dikenalpasti stabil tanpa syarat. Sorotan kaedah yang dicadangkan seperti berikut: Matriks pepenjuru yang diperoleh dari kaedah ini membantu dalam pengiraan penyelesaian yang tepat dan dapat digunakan dengan mudah untuk menyelesaikan PPS dengan kondisi tertentu. Kaedah ini mempunyai kelebihan atas pelbagai kaedah kerana ia menghampiri penyelesaian di semua titik dalam domain dan bukannya pada titik grid. Sumbangan utama tesis ini adalah perkembangan kaedah kuintik splin dan penerapan untuk menyelesaikan persamaan pembezaan separa serakan.

TRIGONOMETRIC B-SPLINE BASED APPROACH FOR SOLVING INITIAL AND BOUNDARY VALUE PROBLEMS OF DISPERSIVE EQUATIONS

ABSTRACT

Various type of numerical methods are developed by employing spline function for solving dispersive PDEs such as finite difference method (FDM), finite element method (FEM) and finite volume method (FVM). Each method inherits certain drawback like complexity, high computational cost and required the trial function or limitation to certain cases. Due to those constraints, FDM incorporated with B-spline function is introduced to solve partial differential equation (PDE). The main aim of this thesis is to explore the accurate and reliable solution to dispersive PDEs. The cubic trigonometric B-spline method (CuTBS), cubic hybrid B-spline method (CuHBS) and two new methods namely quintic trigonometric B-spline method (QuTBS) and hybrid quintic B-spline (QuHBS) method are chosen with the finite difference scheme to solve the third order dispersive PDEs called Modified Regularised Long Wave equation (MRLW), Benjamin -Bona-Mahony-Burgers equation (BBM-Burgers) and Modified Equal Width equation (MEW). The proposed methods produce the numerical solutions that are found to be better or in good compliance with those present methods in literature. Comparison of the maximum error (L_∞) and the Euclidean error (L_2) from the literatures are also done for each example. The performance of the proposed methods are identified to be more accurate than CuTBS and QuTBS method. In order to analyze the stability of the proposed methods, Von-Neumann stability analysis is applied to the linearized schemes. The schemes have been identified to be unconditionally stable. The highlights of the proposed method can be counted as follows: The diagonal matrix obtained from these methods helps in computing accurate solution and can be employed to easily solve PDEs with certain conditions. These methods have an edge over various methods as it approximates the solution at all point in the domain rather than the grid points. The main contribution of this thesis are the development of the quantic splines methods and the applicability to solving dispersive partial differential equations.

CHAPTER 1 : INTRODUCTION

1.1 Introduction to PDEs

Partial Differential Equations (PDEs) are very popular phenomenon occurring in the fields of engineering and mathematical physics. In physics, PDEs are used to describe the phenomenon of wave propagation and heat flow (Bleecker & Csordas, 1992; Farlow, 1993). Thus, they are very essential equations that characterized many scientific and engineering models. A PDE is said to be linear when the power of all dependent variables and every partial derivative which are seen in the equation is equivalent to one, while the dependent variable coefficients and the partial derivative coefficients are either independent variables or constant. But, if any one of the above-mentioned conditions is unsatisfied, it represents a nonlinear equation.

Nonlinear PDEs are present in several physical problems like quantum field theory, fluid dynamics, plasma physics and solid mechanics. Similarly, nonlinear PDEs are also observed in many chemical and biological applications. Nonlinear wave equations have led to many important applications in applied sciences (Ablowitz & Clarkson, 1991; Aminikhah & Alavi, 2013).

Furthermore, nonlinear PDEs is very important and obtaining their solutions are necessary when studying them. Several techniques are used to find their solutions like the numerical or analytical methods. Those methods include the finite element method and variational principle are used to solve the nonlinear PDEs. Furthermore, solving nonlinear PDEs is complicated when the stability of the solutions and their uniqueness are discussed. The complication is because different numerical solutions have been used to solve nonlinear PDEs and a specific solution is needed to satisfy any different conditions (Wazwaz, 2010). For any PDE which regulates the mathematical behaviour of the physical phenomena in the bounded domain, boundary conditions represent the bounded data. Two forms of the boundary conditions are defined below (Wazwaz, 2010)

i. Dirichlet boundary conditions, wherein the function value is mentioned.

ii. The Neumann boundary conditions, where the value of the derivative, which is normal to a boundary is mentioned.

In this study, nonlinear PDEs have been used, especially those in the physical systems like thermodynamics, gas dynamics, relativity, elasticity, neurology, fluid mechanics ecology.

Based on the order of the DEs, the first order nonlinear PDEs are seen to control the nonlinear waves or vibrations. These nonlinear wave motions occur in water waves, gas dynamics, chemical reactions, electrodynamics, chromatography, flood waves, traffic flow and many ecological and biological systems (Wazwaz, 2006). One significant nonlinear phenomenon having no corresponding linear counterpart is known as the breakdown. Determining the solutions for a nonlinear PDE is very important while studying finite time solutions, which results into the formation of some discontinuous shock waves (Wadati, 2001). One striking example includes the supersonic boom that is generated by an aeroplane when it breaks the sound barrier. Likewise, in the linear wave equations, the signals generally propagate alongside the characteristics, however, for a nonlinear equation, the characteristics cross one another, resulting in an onset of the shock waves. Then, the second order PDEs govern the nonlinear diffusion processes, like the population dynamics and the heat flow. The most important and simplest equation for this example is called Burgers' equation. This elementary example can help us to have a glimpse of the nonlinear diffusion processes. As observed, when the viscosity or diffusion in the Burger's equation tends towards zero, the solution for the equation moves the shock waves towards the first order dispersionless limiting equations. Finally, the third order PDEs arise in the study of dispersive wave motion, including water waves, plasma waves, waves in elastic media, and elsewhere. The Korteweg–de Vries equation, which serves as a model for waves in shallow water, waves in plasmas, and elsewhere. This is the basic dispersive model that will be considered. Despite its intrinsic nonlinearity, it supports stable localized traveling wave solutions that, remarkably, maintain their shape even under collision (Wazwaz, 2010). In this study, we have focused on the dispersive model like the BBMB, MRLW and the MEW equations.

1.2 Spline Approach for Solving the Differential Equations

Schoenberg (1946) introduced the spline function for approximating the complex or ill-defined functions or for approximating specific data points with the help of the continuous function. Since then, spline function has been considered essential in several mathematical fields like approximation theory and numerical analysis of the differential and integral equations. Furthermore, it has also proved to be important in various tomography and surgical applications.

A physical analogue was used to understand the mathematical behaviour of the spline, i.e., painter's spline. While planning to pass a smooth curve through several points, the draftsman makes use of a thin and elastic rod which passes through those points. This mechanical spline has a minimal bend or curvature (Kubik & Loon, 1987).

The technique of spline fitting is a popular type of piecewise approximation which uses different forms of the degree m polynomial; or several generalised functions, on intervals where they fit the functions at chosen points, these are called as the control or the node points. These polynomials can change; however, the polynomial derivatives are needed to match the degree $m-1$ at every side of the knots. Finally, boundary conditions are executed at the endpoint of an interval (Islam, 2006)).

Spline functions are used to generate an approximate solution for the differential equations at every point in the period. Ahlberg, Nilson, and Walsh (1967) and Bickley (1968) proposed the use of cubic spline functions to acquire an approximate solution of the two-point Boundary Value Problems (BVPs). Albasiny and Hoskins (1969) used this cubic spline function to get an approximate solution for the BVPs for the second order linear differential equations. Other researchers used the quadratic, cubic and quartic spline functions for solving BVPs which involved the second, third and fourth order linear BVPs (Sakai & Usmani, 1983; Usmani, 1992).

The phrase ‘spline’ was derived based on the flexible device which was used by draftsmen and shipbuilders for drawing a curve through the pre-assigned points (or nodes) such that a continuous curve could be obtained. And also the curvature and the slope of the line were continuous functions. The draftsmen attached a wooden or a metallic strip having weights called as ducks, which could be adjusted for maintaining the strip in the desired shape (Kaur& Jiwari2013).

B-spline on the hand is defined as the spline function with a minimum support regarding to a specific smoothness, degree and domain partition. It is widely used to obtain a numerical solution of linear PDEs. A study by Dosti and Nazemi (2011) showed that the one-dimensional hyperbolic telegraph equation was solved with the help of the cubic B-spline quasi-interpolation method. Demir and Bildik (2012) applied the cubic B-spline finite elements in order to obtain a numerical solution of the heat problem. Goh, Majid, and Ismail (2012) also used the cubic B-spline collocation technique to solve the one-dimensional heat and advection-diffusion equations. The results obtained were much better than previous ones in literature. Aminikhah and Alavi (2013) used the cubic B-spline quasi-interpolation method in order to solve the one-dimensional wave equations using the polar coordinates. Abbas, Majid, Ismail and Rashid (2014) applied the cubic B-spline technique to get solutions for the strongly coupled reaction-diffusion systems.

In many studies, cubic B-spline technique also very helpful in solving nonlinear PDEs. Zarebnia and Parvaz (2013) used the cubic B-spline collocation technique in addition to the finite difference method to solve the Benjamin-Bona-Mahony-Burgers equations. Diyer, Chakrone and Sbibih (2014) applied the cubic B-spline quasi-interpolants to solve the Burger's equation. They included some matrix argument techniques to obtain their result. Karakoç and Zeybek (2016) used the lumped Galerkin approach along with the cubic B-spline function to obtain the solution of the Generalised Equal Width (GEW) wave equations.

The trigonometric spline technique was introduced by Schoenberg (1964) in order to solve some interpolation problems. This method could be applied to the fields of geometric modelling, electronics and approximation theory. The recurrence relation for trigonometric B-spline of the arbitrary order was established by Lyche and Winther (1979). Recently, in the field of geometric modelling, the trigonometric spline technique is being used to develop better CAD/CAM software tools. Based on the observations, the curve modelling is easily handled using the trigonometric spline method, especially when fitting the data onto the spherical objects. The polynomial and trigonometric splines do play vital roles in the development of the Computer Aided Geometric Design (CAGD), particularly in the curve designs (Dube & Yadav 2014).

The trigonometric B-spline collocation technique has garnered a lot of interest among the research community and is used for obtaining numerical solutions of one-dimensional nonlinear PDEs. This type of spline technique shows several geometric characteristics such as smoothness, local support, and capacity to handle local phenomena. All these properties have helped in solving one-dimensional nonlinear PDEs (Birkhoff & Garabedian 1960). Fyfe (1969) observed that this spline method performed much better as compared to the finite difference method due to its flexibility in obtaining a solution for any point within the domain with more accurately. The cubic trigonometric B-spline method stands better in solving mathematical problems particularly the linear and non-linear equations as well as the initial boundary value problem when comparison with the conventional B-spline functions (Nikolis, 2004). Dağ, et al. (2006) used the quintic B-spline Galerkin finite element method to solve the regularized long wave equation (RLW). They also applied the method to solve the time-split RLW equation. Nikolis and Seimenis (2005) applied the cubic trigonometric spline method to solve the nonlinear dynamical systems numerically. They performed an error analysis by using an extended Taylor formula. The method showed a better performance for the periodical functions. Raslan & Hassan (2009) introduced quartic, cubic and quintic B-spline for solving MRLW equations. The methods proved to be very efficient and easy to program and manipulate on digital computers.

Zahra, Ouf and El-Azab (2014) solved the nonlinear parabolic PDEs, known as the Newell-Whitehead Segel equations using different techniques. Their results showed that their cubic trigonometric B-spline approach gives accurate solutions to the linear and the nonlinear equations. On the other hand, cubic B-spline trigonometric technique helped in solving the Burgers' equation, while the finite difference approach helped in discretizing the time and the space derivative. Also, the cubic B-spline trigonometric method acted as the interpolation function within a space dimension. Their studies demonstrated that the trigonometric cubic B-spline approach gives better and more accurate results in comparison to the cubic B-spline technique (Dag, Ersoy, & Kacmaz, 2014).

1.3 Problem Statement

The finite difference scheme is not the only tool for computing approximations to the solution of boundary value problems. There are various approximation techniques which have been examined by many researchers. Spline interpolation method is one of the most effective approximation methods because of its simplicity and practicality (Khalifa, Raslan & Alzubaidi, 2008). The main advantage of using this method is that it can approximate the analytical curve up to certain smoothness. Therefore, the spline method has the flexibility to get the approximation at any point in the domain with more accurate results compared to the usual finite difference method; thus, providing the motivation for this study on examining the accuracy of cubic and quintic uniform B-spline on solving boundary value problems.

Zin, (2016) applied the cubic trigonometric B-spline method, cubic hybrid B-spline method, Quartic trigonometric B-spline and Quartic hybrid B-spline for solving the nonlinear PDEs that contains the nonlinear term from first degree, i.e., ww_x . The methods were performed very well compared to Quartic B-spline method and cubic B-spline method. However, the question is, if using a high degree of trigonometric B-spline and hybrid B-spline, (Quintic trigonometric B-spline method and Quintic hybrid B-spline method) do these methods perform equally well for solving nonlinear PDEs, which contains the nonlinear term from the second degree, i.e., w^2w_x and finally are they are unconditionally stable.

In order to answer the question, the BBMB, MRLW, MEW equations that contain the nonlinear term from second degree, i.e., w^2w_x will be considered as initial-boundary value problems in this thesis. The cubic trigonometric B-spline method, cubic hybrid B-spline method, and two new methods namely quintic trigonometric B-spline and quintic hybrid B-spline will be used to solve the equations that were selected in this thesis. The time derivative is discretised by employing finite difference scheme, while CuTBS, CuHBS, QuTBS and QuHBS as an interpolation function in the space dimension. The stability of the proposed methods is then analysed by using the von Neumann stability analysis method.

1.4 Aims and objective of study

The main objectives of this study are to solve third order nonlinear PDEs using trigonometric B-spline and hybrid B-spline methods. In order to achieve the objective, the following sub-objectives are considered:

a) To apply cubic trigonometric B-spline and cubic hybrid B-spline methods for solving third order nonlinear PDEs namely BBMB equation, MEW equation and MRLW equation.

b) To develop a new method using different B-splines namely Quintic trigonometric B-spline and quintic hybrid B-spline methods for solving third order nonlinear PDEs namely BBMB equation, MEW equation and MRLW equation.

c) To analysis stability of the numerical schemes in (a and b).

d) To compare the proposed methods with the methods available in the literatures.

1.5 Scope Limitation

In order to achieve the first and second objective, two types of basis function will be considered. These functions are the trigonometric B-spline basis function and hybrid B-spline basis function. Those functions will be combined with finite difference approximation for solving BBMB, MRLE and MEW equations. The numerical results will be tabulated, and comparison will be made with those obtained in the literature and the analytical solutions. MATLAB R2012b will be used to perform the calculation and produce the solutions numerically and graphically. The von Neumann stability analysis will be used to investigate the stability of the generated schemes. Finally, the capability of the methods will be demonstrated through some tested problems where the results will be compared with the exact solution and the results from selected problems in the literature.

1.6 Thesis organisation

This thesis contains six chapters. Chapter 1 gives a brief introduction of partial differential equations, the spline approach for solving differential equations, problem statements, aims and objective of study and scope and methodology of the research.

A review of related literature will be provided in Chapter 2. It consists of the histories of spline application on solving a differential equation. In addition, observation of numerical methods which have been used to solve one-dimensional BBMB, MRLW and MEW equation are revealed in this chapter.

In Chapter 3, properties of the B-spline, basis function of the trigonometric B-spline, hybrid basis function of the B-spline, description of trigonometric B-spline method, description of hybrid B-spline method, description of BBMB, description MRLW and description MEW equations will be also discussed in this chapter.

In details, the methodology for solving BBMB equation, MRLW equation and MEW equation by using cubic trigonometric B-spline and cubic hybrid B-spline methods will be discussed in Chapter 4 respectively.

In Chapter 5, quintic trigonometric B-spline and hybrid quintic B-spline method will be used to solve them. In the same chapters, the stability of the linearized equations will be investigated. Finally, the conclusions and suggestions toward future research are presented in Chapter 6.

CHAPTER 2 : LITERATURE REVIEW

2.1 Introduction

In order to resolve the problem of working with higher degree polynomials, the idea of piecewise polynomial come into existence. Instead of using polynomial for the entire domain, the function can be approximated by several polynomials defined over the sub-domains. A polynomial which is presented over a certain domain by means of several polynomials defined over its sub-domains is called a piecewise polynomial. The piecewise polynomial approximation allows us to construct highly accurate approximations, but some approximated functions are not smooth at the connecting point. Even if the polynomial is continuous, it may not be continuously differentiable in the interval of approximation. The basic idea of splines is to construct a piecewise polynomial approximation that is not only interpolate the given data or function values, but it is also smooth i.e. it must be continuously differentiable to some degree (Kaur & Jiwari, 2013). In this chapter, reviews of spline interpolation method, BBMB equation, MRLW equation and MEW equation are discussed particularly.

2.2 Spline: A Brief History

In this section of the thesis, the previous studies will be discussed in the Spline method within the period (1960-2016), and used for each type will be explained.

2.2.1 Spline Interpolation Method

In the engineering and scientific fields which involve several differential equations, the larger equation sets must be solved using numerical methods, based on their accuracy and stability.

The earliest work of using a spline in this list of research papers include to the development method for solving differential equations. In the 1960s Brikhoff and Garabedian used the spline interpolation method for solved second order linear two-point boundary value problems (BVPs). The principle of this method is that a spline function can be randomly defined on a certain domain. Therefore, the spline function is produced on the domain of the problem and the boundary conditions are carried out on it. By reduce the differential equation the resulting system of linear equations and solving its , an approximate numerical solution to the problem is obtained. Results were claimed to be satisfying. Shortly after, the procedure was further modified, developed, generalized and examined. Bickley (1968) suggested the cubic spline. Fyfe (1969) combined the two methods to solve the two-point BVPs. The latter obtained a better approximation to the solution of the problem. To this method, it also produced the solution at any point in the given range, while the finite difference method only obtained the solution at the chosen knots. After that, the author concluded that cubic spline outperformed the existed finite difference method. Another function with a special feature called B-spline was identified in solving a differential equation. Ahlberg and Ito (1975) used the B-spline method in degree three and five to solve two-point boundary value problems. The results obtained were compared to another approach which was used in literature, by examining properties of the established matrices. Al-Said, (1998) uses uniform cubic spline polynomials to derive some new consistency relations. These relations are then used to develop a numerical method for computing smooth approximations to the solution and its first, second as well as third derivatives for a second-order boundary value problem. The present method outperforms other collocations, finite-difference and splines methods of the same order. Rashidinia et al. (2008) suggested cubic spline functions to develop a numerical method for the solution of second-order linear two-point boundary value problems. The resulting linear system of equations has been solved by using a tri-diagonal solver. Convergence of the method is shown through standard convergence analysis. The method is also compared with a finite difference, finite element, B-spline and finite volume methods, which applied to the two-point boundary value problems.

Rashidinia and Sharifi (2015) the collocation method based on quartic B-spline is developed and applied for the two-point boundary value problem in ordinary differential equations. The error analysis and convergence of the presented method are discussed. The method is illustrated by two test examples which verify that the presented method is applicable and considerable accurate. The disadvantages of the previous methods are, they did not extend to solving PDEs system because of needed to effort high and computational cost.

2.2.2 Cubic B-Spline Method

In the last few years, other types of splines function technique have been used increasingly to solve mathematical models. For instance, the cubic B-spline interpolation technique was applied to solve the two-point BVPs for one mathematical model and observed that it performed better than the finite element, finite difference, and the finite volume technique. Hamid, Majid and Ismail, (2010; 2011; 2012) applied numerous cubic B-spline functions for solving two-point BVPs and the resulting accuracy was dependent on the type of the spline applied. They observed that the extended cubic B-spline technique showed a better performance when compared to those of the finite element, finite difference and the finite volume techniques even for very small n $(n=10)$ that representing a partition $(h=b-a/n)$. Furthermore, when they applied the quartic B-spline interpolation process on some examples and compared its results with those of the cubic B-spline and the extended cubic B-spline technique, the quartic and cubic techniques were seen to be more accurate. They found the extended cubic B-spline method is difficult because it is difficult to determine the value of the parameter which depends on the superiority of the method compared to other methods.

Persistent to this study, but in another direction, are results of Goh et al. (2012) where cubic B-spline suggested to solve one-dimensional heat and advection-diffusion equations. The truncation error and the stability of the method were also investigated. This study concluded that results cubic B-spline techniques were more accurate compared to that of FDM for small space-steps. Similarly, the cubic B-spline method approximate numerical solution at each time level whereas FDM only provides discrete solutions. From these studies, it can be deduced that the cubic B-spline method generating better results in the one-dimensional equation with time variable.

Mittal and Tripathi (2014) suggested a collocation-based numerical method to obtain approximate solutions of coupled Burgers' equations. The method employs a collocation of modified cubic B-spline functions. The obtained results were applied in systems of first–order ordinary differential equations (ODEs). The stability of the method was also investigated. The proposed scheme needs less storage space and execution time. This study concluded that cubic B-spline produces more accurate results compared to the methods found in the literature.

There are many other types of Splines, Mohammadi (2015) used the exponential B-spline collocation method to present a numerical algorithm for the time-dependent generalized regularized long wave equation. He semi-discretize the continuous problem by means of the Crank–Nicolson finite difference method in the temporal direction, while using the exponential B-spline collocation method in the spatial direction. The method is shown to be unconditionally stable. This method has high computational costs in comparison to the cubic B-spline, because it is difficult to write programming as it contains a parameter that controls the preference of the method

The disadvantages of the previous methods did not extend to solve nonlinear PDEs systems, because of high computational costs, in comparison to PDEs equations and nonlinear PDEs equations.

2.2.3 Trigonometric B-Spline Method

The trigonometric B-splines are called as the non-polynomial B-spline base as they consist of the sine function. They have several applications in the field of engineering or science. Furthermore, this technique is also applied as the approximate function for developing a numerical method consisting of the nonlinear partial differential equations (Nikolis & Seimenis, 2005).

Hamid et al. (2010) applied the cubic trigonometric B-spline method for solving the linear two-point BVPs of second order. They tested this method on four different problems and then compared their results with the cubic B-spline technique. Then, Dag et al. (2014) used the cubic trigonometric B-spline method for solving the Burger's equation based on the Crank-Nicolson approach for solving the time discretisation problem. They also used this CuTBS function to discretize the spatial variable. This technique showed a better efficiency for the solution of the equation.

In one study, Zahra et al. (2014) used cubic B-spline technique for numerically solving the Newell-Whitehead Segel equation. They observed that the numerical scheme was unconditionally stable when they used the linear von Neumann stability analysis. After computing the maximum numerical errors, their results showed that the trigonometric cubic B-spline technique performed better than the cubic B-spline process. Thus, this indicated that several linear or nonlinear PDEs could be solved using this technique.

Furthermore, Zin et al. (2014) also applied this approach while including a finite difference scheme to determine an approximate solution for the nonlinear Klien -Gordon equation using the Dirichlet boundary conditions. They used the trigonometric cubic B-spline method for interpolating a solution within the space dimensions. They also used the central difference for discretising the time derivatives. They determined this method's performance by calculating the absolute errors and the L_∞ error norm at various times. They showed that their results were in a good agreement with the exact solution.

Nazir, Abbas, Ismail, and Majid (2015) applied the cubic trigonometric B-spline method for solving the one-dimension second order hyperbolic telegraph equations having an initial condition and the Dirichlet and Neumann boundary conditions. They also used the finite difference technique to discretize the time derivative whereas the trigonometric cubic B-spline function was used to interpolate the space derivative. Furthermore, they used the von Neumann process to determine the unconditional stability. All in all, they have tested five examples, after computing L_2, L_∞ and the Root Mean Square (RMS) error, and their results showed a good performance in terms of accuracy and in good agreement with the exact solution as compared to other studies (Dosti & Nazemi, 2012).

Ersoy and Dag (2016) also used the cubic trigonometric B-spline technique to solve the nonlinear Burger's equation. In the study, 3 test examples were used and the results were compared to other studies (Khater, Temsah, & Hassan, 2008; Mittal & Arora, 2011; Mittal & Tripathi, 2014). They stated that the cubic trigonometric B-spline-based numerical approach could reliably solve the PDEs.

Karakoç and Zeybek (2016) used a collocation method that was based on the cubic trigonometric B-spline technique combined with Crank-Nicolson technique to solve the nonlinear Klein-Gordon equation. They decreased the order of the equation resulting into a coupled set of nonlinear partial differentials for carrying out a time integration using the Crank-Nicolson implicit technique and were fully discretised with the help of the collocation approach. This method helped in propagating a single positive solitary and simulating the movement of the wave. Additionally, it also helped in estimating the energy, and momentum of the conservation laws along with estimating the error occurring between the analytical and the numerical solutions using the discrete maximum norm. All their results proved the accuracy and validity of their method. The first disadvantage of the previous method it was found that cubic trigonometric B-spline produces more accurate results if the exact solution was a trigonometric function. Second, of the disadvantages, the method cited above indicates applied for solving linear two-point BVPs of the second order, PDEs and nonlinear PDEs that contain the first nonlinear term ww_x . In this thesis plan to apply it to solving dispersive equations (BBMB, MRLW and MEW equations) that contain second order of nonlinear term w^2w_x .

2.3 Benjamin-Bona-Mahony-Burger's (BBMB) Equation

The nonlinear equations are very important in the fields of engineering, applied physics and mathematics (Chen, Li, & Zhang, 2005). In each of these equations, the parameters are seen to vary based on different factors. In the case of the solitary wave using the general Benjamin - Bona-Mahony-Burger's (BBMB) equation, there is a good balance present between the nonlinear and the dispersive effects; however, it does not consider the dissipation. Physically, an equation having a dissipative term of $-\alpha u_{xx}$ is suggested if one desires a suitable predictive power, for instance, a model can fit the above phenomenon and has been used by several researchers (Tari, & Ganji, 2007).

Al-Khaled, Momani, and Alawneh (2005) applied the Adomain decomposition technique for numerically solving the BBMB equations. According to the three tested problems, the researchers concluded that the method was distinctive as it avoided many complexities and extensive computational work when compared to many other methods.

Additionally, Tari and Ganji (2007) used two techniques of Homotopy Perturbation (HP) and Variational Iteration (VI) for solving a special type of the general nonlinear BBMB equation. Their results indicated that their numerical solution was in good agreement with the exact solution. The approximated solution was also seen to be more tangible as the value of t was higher than 0.4. Numerous iterations and a higher order for p were needed for a better accuracy, using VI and HP methods, respectively. These numbers of iterations and the order of p were dependent on the nonlinear equation term. VI and the HP methods have many advantages improving the exact solution. Therefore, they can be efficiently used to resolve the problems occurring in calculating the exact solution.

Omrani and Ayadi (2008) applied a Crank-Nicolson- type finite difference approach for solving similar equations. They determined the distinctiveness and the stability of their method after using the discrete energy technique. They also studied the specific case of $\alpha = 0$, i.e., BBM equation and showed the energy conservation.

Zarebnia and Parvaz (2013) used the cubic B-spline collocation approach for solving nonlinear BBMB equations. They also studied the stability of the method using the von Neumann stability analysis. Moreover, they also estimated the L_2 and L_∞ error norms based on two examples and compared them to their numerical results having exact solutions for various time points and values (h). But they did not compare the results with other methods to check accuracy and efficiency of their method.

Yin and Piao (2013) applied the quadratic B-spline finite element approach for spatial variables combined with the Newton's method to solve similar equations. They initially used the quadratic B-spline finite element approach to convert the BBMB equation to the finite set containing nonlinear ordinary differential equations, after which they used the Newton's technique for the time variable. The results that were obtained for various viscosity were compared with the exact solution. However, they did not calculate the absolute error, the maximum error and the Euclidean error to prove its accuracy and its efficiency.

Arora, Mittal, and Singh (2014) used the quartic B-spline collocation approach to determine the numerical solution for their research equations. They initially tested four problems to evaluate the performance of their method and to estimate the error norms for various time points and then compared their results to other studies Omrani and Ayadi (2008). They estimated the convergence rate to be second order convergent. Their method was very accurate and could provide good solutions for many parameters. However, they observed the error decreased as time increased, and the linear stability analysis was used to prove the unconditional stability of the method.

2.4 Modified Regularised Long Wave (MRLW) Equation

Peregrine (1966) firstly described the Modified Regularised Long Wave (MRLW) equation to describe the formation of the undular bore, explaining that the wave motion could be used as the alternative to the common Korteweg-de Vries equation (Hassan & Alamery, 2009). In 1985, the MRLW equation was used as the model for a unidirectional propagation of water waves with small amplitudes and a longer wavelength. As it is a very important nonlinear dispersive wave equation, it is used in many areas like the ion-acoustic waves used in plasma, phonon packets within crystals, pressure waves seen in a liquid-gas bubble mixture and in a tube and transverse waves noted in shallow waters (Karakoc & Geyikli, 2013). This equation must be accurately solved as it could help to understand the nonlinear wave behaviour. The MRLW or the nonlinear dispersive wave equation is used to model the waves in shallow waters. The major characteristics of the equation were taken into consideration along with the fact that a solitary wave moved in a single direction with a steady speed and maintained its shape. MRLW equation is based on the RLW equation (Dağ, Irk, & Sarı, 2013).

Several researchers determined the solution for this equation using various techniques. In their studies, (Dağ et al., 2004; 2006; 2013) used numerous techniques. Initially, in 2004, they applied the cubic B-spline collocation technique to solve the RLW equation, and they also applied the form described by Rubin and Graves (1975) to linearize the nonlinear terms within the equation. They tested the performance of their method after calculating the solitary wave solution and then comparing their results to the analytical results and other methods from literature for time ($t = 20$). They showed that their method could help in accurately and reliably solving the RLW equation ($U_t + U_x + \varepsilon U U_x - \mu U_{xxt} = 0$). Then in 2006, they used the quintic B-spline Galerkin finite element approach to solve this equation. They proved the accuracy of Galerkin method by using the quintic B-spline approach to express the approximate functions within the finite element technique. They used this method to solve the time- spit RLW equation $((U - \mu U_{xx})_t + 2\varepsilon U U_x = 0$ and $(U - \mu U_{xx}) + 2U = 0)$. They studied the motion of the solitary wave and the behaviour of the interaction between two waves after applying the conservation law (Dağ et al., 2006). They further compared their results to those published earlier (Zaki, 2001). Their results indicated that the numerical solution was fairly accurate and the quintic B-spline functions could be applied to solve higher orders PDEs. Furthermore, Dağ et al. (2013) used the extended -based collocation approach to solve the MRLW. In their extended cubic B-spline approach, they introduced a free parameter in the cubic B-spline and determined its value. Thereafter, they noted that the final result after using this algorithm was similar to that obtained after using CuBS (Cubic B-Spline method). Thus, their result revealed that their method was appropriate for solving the MRLW equation and showed better results than the CuBS method.

In their study, Hassan and Alamery (2009) used a higher order B-spline technique (sextic B-spline collocation approach) to solve the MRLW equation. Therefore, the first order systems could be solved using the fourth order Runge-Kutta technique, which is a very powerful technique for solving these problems. They evaluated three invariants of motion in studying the algorithm's conservation properties. Although they made the comparison only at conservation law, results showed that the sextic B-spline function could determine an approximate numerical solution for the PDEs of higher order derivatives.

Karakoc, Geyikli, and Bashan (2013) and Karakoc and Geyikli (2013) also used two novel methods to solve the MRLW equation, i.e., a subdomain finite element technique that was based on the quartic B-spline function and the Petrov-Galerkin approach which used the quadratic weight functions along with a cubic B-spline finite element. Both methods examined the solitary wave motion wherein the analytical solutions were known and also studied the interaction between two and three solitary waves as well as the

Maxwellian's initial condition. After estimating the error, L_2 and L_∞, they proved the efficiency and accuracy of their methods in solving the MRLW equation. They found the accuracy of methods decrease as the time increases.

On the other hand, Yusuf (2015) used the Lumped Galerkin approach which was based on the cubic B-spline finite elements for solving the MRLW equation. They investigated the conservation properties of their scheme by determining the invariant quantities of mass, momentum and energy. They found the percentage of the relative error of the mass, momentum and energy which is calculated for $t = 0$ not remain static during the computer run. They also calculated the error norms of L_2 and L_∞ only for single solitary wave for comparison with Karakoc and Geyikli (2013) at $t = 10$.Their results showed that their proposed technique was more accurate as compared to the method proposed by Karakoc and Geyikli (2013).

2.5 Modified Equal Width (MEW) Equation

The Equal Width (EW) wave-based Modified Equal Width (MEW) wave equation was first proposed by Morrison, Meiss and Cary (1984) as the model PDE for simulating and propagating the waves in a nonlinear media using dispersion processes. The MEW equation was similar to the altered Korteweg-de Vries equation and the MRLW equation. All these modified equations were nonlinear wave equations having cubic nonlinearities, with solitary wave solutions, and were in form of either pulses or packets (Karakoç & Geyikli, 2012).