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# STABILISATION OF FRACTIONAL ORDER DYNAMICAL CONTROL SYSTEMS BASED ON BACKSTEPPING METHOD



by

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## LIST OF ABBREVIATIONS

| | |
|---|---|
| IPDEs | Integer Order Partial Differential Equations |
| DPSs | Distributed Parameter Systems |
| ODEs | Ordinary Differential Equations |
| PIDEs | Partial Integro-Differential Equations |
| FPDEs | Fractional Order Partial Differential Equations |
| FPIDEs | Fractional Order Partial Integro-Differential Equations |
| FOSs | Fractional Order Systems |
| FOLSs | Fractional Order Linear Systems |
| LMI | Linear Matrix Inequality |
| FONSs | Fractional Order Nonlinear Systems |
| MRAC | Model Reference Adaptive Control |
| FOABC | Fractional Order Adaptive Backstepping Control |
| SMC | Sliding Mode Control |
| LQR | Linear Quadratic Regulator |
| BTCS | Backward – Time Central - Space |

## LIST OF SYMBOLS

| | |
|---|---|
| ${}_{0}^{c}D_{t}^{q}$ | Caputo fractional order derivative |
| ${}_{0}I_{t}^{q}$ | Riemann-Liouville fractional order integral |
| $\Gamma$ | Gamma function |
| max | Largest value of the function |
| $L_{\infty}$ | The space of essentially bounded functions |
| $E_{q}$ | The Mittag-Leffler function in one parameter |
| sup | The least upper bound |
| $c^{1}$ | The set of all continuously differentiable functions |
| $H^{1}$ | The usual Sobolev space |
| $L^{2}$ | The space of square integrable functions |
| $c^{2}$ | The set of all twice continuously differentiable functions |
| $\Re$ | The set of real numbers |
| $\Re^{n}$ | *n*-dimensional space |
| spec (A) | Spectrum of A |
| arg (A) | Argument of A |
| c | The set of all continuous functions |
| $\lVert\cdot\rVert_{\infty}$ | $L_{\infty}$ norm |
| $\lVert\cdot\rVert$ | $L^{2}$ norm |
| $\lVert\cdot\rVert_{H^{1}}$ | $H^{1}$ norm |
| $s_{x}$ | The first partial derivative of $s$ with respect to the space |
| $s_{xx}$ | The second partial derivative of $s$ with respect to the space |
| $s_{x}^{\text{dis}}$ | The discretisation of first partial derivative of $s$ with respect to the space |
| $\hat{s}$ | Estimator of $s$ |
| $m$ | Number of points used in discretisation |
| N | The set of integer numbers |

**Penstabilan Sistem Kawalan Dinamik Berperingkat Pecahan Berdasarkan Kaedah Langkah Belakang**

## ABSTRAK

Pada masa kini, kawalan sempadan bagi persamaan pembezaan separa berperingkat integer (PPSI) telah menjadi satu bidang penyelidikan yang penting. Ini disebabkan oleh peningkatan permintaan terhadap kawalan berketepatan tinggi dalam kebanyakan sistem mekanikal. Secara umumnya, fenomena universal boleh dimodelkan dengan lebih tepat menggunakan persamaan pembezaan separa berperingkat pecahan (PPSP). Oleh yang demikian, terdapat minat yang semakin meningkat dalam menyelidiki penyelesaian dan sifatnya. Berbanding dengan kajian mengenai kawalan PPSI, kajian ke atas kawalan PPSP adalah sangat terhad dan perlu diterokai. Tesis ini memberi tumpuan kepada membangunkan prosedur-prosedur yang sistematik berdasarkan kaedah langkah belakang untuk kestabilan sistem PPSP dan persamaan pembezaan integro separa berperingkat pecahan (PPISP). Kestabilan PPSP dan PPISP dicapai melalui dua pendekatan. Pendekatan pertama menggunakan kaedah langkah belakang berdimensi terhingga melalui rekabentuk transformasi koordinat. Transformasi ini mempunyai bentuk hubungan rekursif dengan bilangan lelaran tak terhingga. Daripada simulasi berangka, keputusan menunjukkan bahawa teras menumpu kepada fungsi terbatas tetapi tidak selanjar. Pendekatan kedua pula menggunakan kaedah langkah belakang berdimensi tak terhingga. Dalam pendekatan ini, satu penjelmaan kamiran memetakan sistem PPISP kepada sistem sasaran stabil Mittag-Leffler terpilih yang sesuai. Terasnya ditakrifkan melalui penyelesaian PPIS hiperbolik. Daripada simulasi berangka, keputusan menunjukkan teras bukan sahaja terbatas, tetapi selanjar dan boleh beza dua kali. Selain itu, kaedah langkah belakang digunakan untuk merekabentuk pemerhati bagi anggaran ruang sistem PPISP linear dengan syarat sempadan Dirichlet dan Neumann. Dua ketetapan diberikan. Ketetapan pertama ialah anti-dihimpunkan yang mana sensor dan penggerak ditempatkan pada hujung yang bertentangan. Ketetapan kedua pula ialah dihimpunkan yang mana sensor dan penggerak ditempatkan pada hujung yang sama. Keputusan menunjukkan bahawa kedua-dua pendekatan menghasilkan prestasi yang memuaskan dalam menangani sistem PPSP dan PPISP linear yang tidak stabil. Seterusnya, kaedah langkah belakang separa-pengdiskretan diperkenalkan untuk mencari pengawal sempadan bagi sistem PPSP tak linear. Untuk tujuan ini, tiga kes sistem PPSP tak linear dipertimbangkan iaitu sistem PPSP tak linear dengan terbitan pecahan terhadap ruang, sistem PPSP tak linear dengan terbitan pecahan terhadap masa dan sistem PPSP tak linear dengan terbitan pecahan terhadap ruang dan masa. Kesemua sistem PPSP tak linear akan ditransformasikan menjadi gelung tertutup yang sama dan bentuk analitik petua kawalan maklumbalas direkabentuk. Untuk kes yang pertama, penumpuan bagi sistem gelung tertutup dijamin oleh prinsip invarians LaSalle manakala kes yang kedua dan ketiga dijamin oleh kestabilan Mittag-Leffler. Daripada simulasi berangka, keputusan menunjukkan kaedah langkah belakang separa-pengdiskretan sangat berguna untuk menstabilkan sistem PPSP tak linear. Walau bagaimanapun, pengiraan simbolik kawalan maya menjadi lebih mahal apabila nilai saiz langkah pengdiskretan menghampiri sifar.

**Stabilisation of Fractional Order Dynamical Control Systems Based on Backstepping Method**

## ABSTRACT

Nowadays, boundary control of integer order partial differential equations (IPDEs) has become an important research area. This is due to the increasing demand on high-precision control of many mechanical systems. In general, the universal phenomenon can be modeled more accurately using fractional order partial differential equations (FPDEs). Therefore, there has been a growing interest in investigating the solution and properties of FPDEs. Compared with the study on control of IPDEs, the results on control of FPDEs are very limited and need to be explored. This thesis focuses on the development of systematic procedures based on backstepping method for stabilisation of FPDE and fractional order partial integro differential equation (FPIDE) systems. Stabilisation for linear FPDE and FPIDE systems is achieved by using two approaches. The first approach uses a finite dimensional backstepping method through the design of coordinate transformations. These transformations have the form of recursive relationships with infinite number of iterations. From numerical simulation, the result showed that the kernel converges to a bounded but possibly discontinuous function. The second approach uses the infinite dimensional backstepping method. In this approach, an integral transformation maps the FPIDE system to a suitably selected Mittag-Leffler stable target system. The kernel is defined by the solution of the kernel hyperbolic partial integro differential equation (PIDE). From numerical simulation, the result showed that the kernel is not only bounded but twice continuously differentiable function. In addition, the infinite dimensional backstepping method is used to design observer for state estimation of linear FPIDE with Dirichlet and Neumann boundary conditions. Two setups are provided. The first setup is anti-collocated when sensor and actuator are placed at the opposite ends. The second setup is collocated when sensor and actuator are placed at the same end. The results showed that both approaches yield satisfactory performance in dealing with unstable linear FPDE and FPIDE systems. Moreover, the semi-discretised backstepping method is introduced to find the boundary controller of nonlinear FPDE system. For this purpose, three cases of nonlinear FPDE system are considered, which are nonlinear FPDE system with space fractional derivative, nonlinear FPDE system with time fractional derivative and nonlinear FPDE system with space and time fractional derivatives. All cases of nonlinear FPDE system are transformed into equivalent stable closed loop and the analytic forms of feedback control laws are designed. For the first case, the convergence of the closed loop system is guaranteed by LaSalle's invariance principle while for the other two cases the convergence is guaranteed by Mittag-Leffler stability. From numerical simulation, the results showed that the proposed semi-discretised backstepping method is powerful to stabilise the nonlinear FPDE system. However, the symbolic calculation of the virtual control becomes more expensive when the value of the discretisation step size approaches to zero.

# CHAPTER 1 : INTRODUCTION

## 1.1 Background

Fractional calculus is a generalisation of integer order integral and differential calculus to any arbitrary real or even complex order. This generalisation extends a traditional definition of integration and differentiation to non-integer order and joins them into one definition where the operator depends on the order sign. The popularity of this subject has increased during the last decades in several fields of science and engineering. Development of algorithms and methods for solving fractional equations and special functions allows the fractional calculus to become a very useful tool for precise description of real-world phenomena.

The concept of fractional calculus was introduced towards the ending of 17th century. Gottfried Wilhelm Leibniz (1646–1716) wrote the reply letter to Marquis de L'Hˆopital (1661–1704) answering to the question about the notation $\frac{d^n y}{dx^n}$ for $n = \frac{1}{2}$, where he presented the following conclusion "This is an apparent paradox where, beneficial consequences will be drawn one day" (Sarwas, 2012).

This letter had become a motivation for future generations of mathematicians, such as: Euler, Liouville, Riemann, Grunwald, Letnikov, and many more, who, till the end of the 19th century, formed the basis of the fractional calculus. However, the first monograph was published only in 1974 (Oldham & Spanier, 1974). More information on the subject can be found in work by the following researchers: Anastassiou (2011), Baleanu, Machado and Luo (2012), Caponetto (2010), Kaczorek (2011), and Monje, Chen, Vinagre, Xue and Feliu-Batlle (2010).

In the beginning, theories involving differential, integral, as well as integro-differential equations were considered as the domain of mathematical physics along with their overviews and extensions in the involved variables. In the second half of the 20th century, parallel to the development of computers and algorithms, this calculus has spread out to the engineering field where it became a very powerful and useful tool in many applications such as in fluid flow, mechanics, rheology, control theory of dynamical systems, viscoelasticity, optics and signal processing and many more (Sarwas, 2012).

Application of fractional order calculus in the control theory has forced the generalisation of this domain for non–integer order. Many researchers have described the definitions and properties of the continuous fractional order systems (Hartley & Lorenzo, 2002; Kaczorek, 2009; Quintana, Ramos & Nuez, 2006). Issues concerning fractional system identification can be found by Le Lay, Oustaloup, Levron and Trigeassou (1998).



In many applications, such as diffusion and structural vibrations, the physical quantity of interest depends on both position and time. These systems are modeled by IPDEs and the solution evolves on an infinite-dimensional Hilbert space. For this reason, these systems are often called infinite-dimensional systems. In contrast, the state of a system modeled by an ordinary differential equations (ODEs) evolves on a finite-dimensional system, such as $\Re^n$, and these systems are called finite-dimensional. Since the solution of the IPDE reflects the distribution in space of a physical quantity such as the temperature of a rod or the deflection of a beam, these systems are often also called distributed parameter systems (DPSs). The purpose of controller design for infinite-dimensional systems is similar to that for finite-dimensional systems. Every controlled system must of course be stable.

Control of IPDEs involves two types of settings, which are reliant on the location of both actuators and sensors in the domain control. The actuation would then enter the domain and be distributed in the domain and boundary control. The actuation and sensing were applied only through the boundary conditions. Usually, boundary control is normally known to be more realistic because actuation and sensing are nonintrusive. Besides, it is also thought considered to be the harder problem, as the input operator and the output operator are unbounded operators. As a result of the greater mathematical difficulty, lesser methods were developed over the years for boundary control problems of IPDEs.

The backstepping control is one category of control approaches that has gained a considerable attention in the case of controlling parametric nonlinear strict feedback systems. It is based on the control approach which is a recursive design technique that breaks down the control design for the full system into a sequence of lower-order subsystems (Rashad, Aboudonia & El-Badawy, 2016). Backstepping is unlike any of the methods previously developed in literatures for controlling ODEs and IPDEs. It differs from optimal control methods in that it sacrifices optimality (though it can achieve a form of "inverse optimality") for the sake of avoiding the operator of Riccati equations, which are very hard to solve for infinite or high dimensional systems, such as

IPDEs. Backstepping is also different from pole placement methods, because even though its objective is the stabilisation of the system, which is also the same objective of the pole placement methods. In addition, backstepping does not pursue precise assignment of even a finite subset of the IPDE's eigenvalues (Krstic, Kanellakopoulos & Kokotovic, 1995).

## 1.2 Problem Statement



The boundary stabilisation for an integer order unstable heat system has been solved by many researchers using the boundary control law, that is known as backstepping control law, which is basically formed as an integral operator with an identified continuous kernel function (Krstic & Smyshlyaev, 2008b; Smyshlyaev & Krstic, 2010; Zhou & Guo, 2013). Nevertheless, according to previous work, there are only limited attempts on the method of boundary feedback stabilisation to deal with the unstable FPDEs and FPIDEs. Based on the numerical simulation techniques, the boundary stabilisation of a one-dimensional fractional diffusion-wave equation was studied by Liang, Chen and Fullmer (2004) and the boundary control of a Caputo fractional wave equation via a fractional-order boundary controller was presented.

It is confirmed that many real-world life systems can be characterised by applying the concepts of fractional order (Torvik & Bagley, 1984). This is why the fractional-order models are superior as compared to the integer order models. In real world, the continuous time random walk can be one useful technique to describe the phenomena of the diffusion process that is essentially distributed (Cartea & del-Castillo-Negrete, 2007). For instant, particles perform sub-diffusion when they presume to jump during fixed time intervals with a random waiting time. Furthermore, the time fractional order diffusion system can be used to characterise them.

For the operation of a fractional control system, the knowledge of states of the fractional system is important. In most cases it is not possible to have full information of the fractional system's states due to the fact that not all of the variables can be measured. Installing all the necessary sensors may not be physically possible or the costs may become prohibitive. In such a case, the states can be estimated using fractional observer. However, in FPDE systems, the states, inputs, and outputs depend on some spatial variable. This dependence, along with additional aspects such as the boundary conditions, increases the complexity of the state estimation problem and of the design method.

Motivated by the arguments above, this thesis focuses on open-loop unstable linear FPDE & FPIDE systems, fractional output feedback problem and open-loop unstable nonlinear FPDE system and delivers feedback laws of acceptable complexity which solve the fractional stabilisation problem.



## 1.3 Research Objectives

This thesis is driven based on the observation of the FPDE and FPIDE systems, which are involved in many applications. Generally, this thesis is aimed to develop systematic procedures based on backstepping method with guaranteed stability in designing controller for fractional order dynamical systems. To be more precise, the objectives of this study are as follows:

i. To stabilise linear FPDE and FPIDE systems using finite dimensional backstepping method.

ii. To stabilise linear FPIDE system using infinite dimensional backstepping method.

iii. To design a Mittag-Leffler convergent observer for linear FPIDE system with both anti-collocated and collocated setups.

iv. To stabilise nonlinear FPDE system with space fractional derivative, nonlinear FPDE system with time fractional derivative and nonlinear FPDE system with space and time fractional derivatives using semi-discretised backstepping method.

## 1.4 Contributions of Thesis

The following are the main contributions of this thesis.

i. Finite dimensional backstepping coordinate transformation is used to control the linear (FPDE & FPIDE) systems such that the design procedure can handle fractional order systems with an arbitrary finite number of open loop unstable eigenvalues. Stabilisation is achieved through the design of coordinate transformations that have the form of fractional recursive relationships.

ii. Backstepping boundary controller is applied to control the linear FPIDE system with two types (Dirichlet & Neumann) boundary actuation avoiding spatial discretisation required in previous contribution. By designing an invertible coordinate transformation, the system under consideration was transformed into a Mittag-Leffler stability linear system while the boundary stabilisation problem was converted into a problem of solving a linear hyperbolic PIDE.

iii. The output feedback problem for linear FPIDE system with both anti-collocated and collocated actuator/sensors pairs (where the boundary conditions is Dirichlet boundary conditions or Neumann boundary conditions) is regarded as a problem in terms of designing an invertible coordinate transformation of the fractional order observer error system into a Mittag-Leffler stable fractional order target system. Output injection function is shown to satisfy a well-posed hyperbolic PIDE that is closely related to the hyperbolic PIDE governing backstepping control gain for the state feedback problem.

iv. The discretised fractional order backstepping technique has been developed for stabilising the nonlinear FPDE system with two types of fractional derivatives (Caputo and Grünwald-Letnikov) definitions. With this technique, an effective boundary controller can be designed for nonlinear FPDE system. The designs procedure consist of three stages are constructed such that the boundary controller can always be constructed with appropriate choices of some design parameters. The analytical forms of control law by using two types of fractional derivatives are presented.



### 1.5 Significant of Contributions

The results of this study may have a significant impact on proving systematic procedures based on backstepping method for fractional order dynamical systems. It is understood that the work presented in this thesis is dedicated to the fundamental academic exploration of boundary control of fractional order systems. Thus, the focus is given to the development of the control method. In addition, this study is focused on the FPDE and FPIDE systems, which cover large classes of flexible string and beam systems in mechanical engineering. It would be a future research topic to extend the proposed control design methods to fractional order systems in other forms.

## 1.6 Thesis Outline

The organisation of this thesis is as follows:

i. Chapter 2 contains literature survey about the fractional order dynamic control systems, fractional Lyapunov function, and the boundary control of IPDEs.

ii. In Chapter 3, a family of stabilising boundary feedback control laws for a class of linear (FPDE & FPIDE) systems are presented. Stabilisation is achieved through the design of coordinate transformations that have the form of recursive relationships. The fundamental difficulty of such transformations is that the recursion has an infinite number of iterations. The existence of a non-smooth but bounded gain kernel was proved.

iii. In Chapter 4, a problem of boundary stabilisation of linear FPIDE system is considered using the infinite dimensional backstepping method avoiding any spatial discretisation required in Chapter 3. The backstepping based observer design for Mittag-Lefller convergent state estimation of FPIDE system is provided for anti-collocated and collocated setup with (Dirichlet/Neumann) boundary conditions.

iv. Chapter 5 focuses on the application of fractional backstepping control scheme for nonlinear FPDE system. Two types of fractional derivatives are considered Caputo and the Grünwald-Letnikov fractional derivatives. Therefore, obtaining highly accurate approximations for these derivatives are of a great importance. The discretised approach for the space variable is used to transform the nonlinear FPDE system into a system of nonlinear fractional differential equations. The convergence of the closed loop system is guaranteed in the sense of LaSalle's invariance principle and Mittag-Leffler stability.

v. The conclusions and recommendations for future work are presented in Chapter 6.

## 1.7 Summary

This chapter introduced FPDE and FPIDE systems and analysed the problem based on previous work. The objectives and contributions of the study were also briefly explained in this chapter. The method for stabilisation FPDE and FPIDE systems was also identified.

# CHAPTER 2 : LITERATURE REVIEW

## 2.1 Introduction

During the last few decades, there has been a great effort from researchers to include fractional order systems in control community. This could be due to two major factors: first, many engineering applications cannot be well described by the ordinary integer order systems (Sheng, Chen & Qiu, 2012; Yuan & Sun, 2013), and second, the fractional controller has shown that it has more potential and design freedom compared to a standard integer order controller (Pan & Das, 2013; Tang, Zhang, Zhang, Zhao & Guan, 2013).

The theory of the stability, controllability and observability of a fractional order state space system has been reported in 1996 by Matignon. His work was extended into combining fractional calculus to modern control theory (Matignon, 1996; Matignon & d'Andréa-Novel, 1996). More recently, researchers have been applying fractional calculus to control systems, which resulted in increased effectiveness in systems such as a robust control system, and a nonlinear control system.

In mathematics, the stability theory addresses the convergences of solutions of differential or difference equations and of trajectories of dynamical systems under small perturbations of initial conditions. Same as classical differential or difference equations a lot of stress has been given to the stability and stabilisation of the systems represented by fractional order differential equations.

## 2.2 Stability of Fractional Order System

The stabilisation problem in control systems is quite well reported and established in literature. Incidentally, this includes the fractional order control systems, and has been the focus of many work by researchers. Many of the reported results pertaining to the fractional order systems (FOSs) stabilisation focuses on fractional order linear systems (FOLSs). The Matignon's theorem makes up one of the first stability criterion of the FOLS (Matignon, 1996). The linear matrix inequality (LMI) representations were suggested by Sabatier, Agrawal and Machado (2007), while its requirements were analysed by Li and Wang (2012). In the context of LMI conditions, the pseudo-state feedback stabilisation of deterministic FOLSs was addressed by Zhang, Liu, Feng and Wang (2013). The majority of robust stabilisation results were reported by Farges, Fadiga and Sabatier (2013) and Padula, Alcántara, Vilanova and Visioli (2013). Early robust stabilisation results can be inferred from Lu, Chen and Chen (2013). Moreover, $H_\infty$ control problems of FOSs were suggested by Shen and Lam (2014). However, real FOSs always have many non-linear structures (Ding, Qi & Wang, 2015).

In related, there has been formed a new class of nonlinear systems called Fractional-order non-linear systems (FONSs) has been recently reported by Baleanu et al. (2012). Lyapunov's direct method provides a way to analyse the stability of a system without explicitly solving the differential equations the method generalises the idea, which shows that the system is stable if there are some Lyapunov function candidates for the system. The Lyapunov direct method is a sufficient condition to show the stability of systems, which means the system may still be stable even one cannot find a