NEW APPROXIMATION METHODS BASED ON FUZZY TRANSFORM FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

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by

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

$\mathbb{R}$	Real numbers
$\mathbb{R}^+$	Positive real numbers
$\mathbb{N}$	Natural numbers (including zero number)
$\mathbb{N}^+$	Positive natural numbers
$\mathbb{Z}$	Integer numbers
$F[f]$	Direct fuzzy transform with respect to function f
$\hat{f}$	Inverse fuzzy transform
K_{T^m}	Power of the triangular (shaped) generating functions
K_{C^m}	Power of the raised cosine generating function
$(.)^T$	Transposition

LIST OF ABBREVIATIONS

FzT	Fuzzy transform
ODEs	Ordinary differential equations
IVPs	Initial value problems
Euler-FT	Fuzzy transform based Euler method
Mid-FT	Fuzzy transform based Mid-point rule
NIM	New iterative method
MSE	Mean square error

Kaedah-kaedah Penghampiran Baru Berdasarkan Transformasi Kabur untuk Menyelesaikan Persamaan Pembezaan Biasa

ABSTRAK

Masalah-masalah dunia sebenar dalam sains dan kejuruteraan dimodelkan menggunakan persamaan-persaman pembezaan. Dalam kebanyakan kes, persamaan-persamaan pembezaan tidak boleh diselesaikan secara analitik, maka kaedah-kaedah berangka digunakan. Menjelang masa berlalu, pengkaji menyedari pendekatan kabur untuk penyelesaian berangka bagi persamaan-persamaan pembezaan menjadi penting. Salah satu pendekatan kabur yang telah diperkenalkan dalam literatur ialah transformasi kabur (TKb). Terdapat minat yang semakin meningkat dalam penyiasatan sifat-sifat sesekat-sesekat kabur. Dalam kajian ini, perwakilan-perwakilan baru suatu fungsi-fungsi asas telah diperkenalkan. Hal ini tercapai dengan memperkenalkan sesekat-sesekat kabur sekata teritlak baru yang dipanggil kuasa oleh sesekat-sesekat kabur sekata teritlak segitiga dan kosinus terbangkit. Ciri-ciri asas suatu sesekat-sesekat kabur sekata teritlak baru telah diperkenalkan. Selanjutnya, bentuk lebih mudah suatu TKb diberikan bersama sebahagian penyelesaian-penyelesaian asasnya. Teorem-teorem dan lemma-lemma yang baru telah diperkenalkan dan dibuktikan secara matematik. Kemudian, sesekat-sesekat kabur sekata teritlak baru digunakan dalam tiga kaedah penghampiran berdasarkan kepada TKb bagi menyelesaikan masalah-masalah Cauchy. Kaedah penghampiran yang pertama menggunakan peraturan Trapezium dengan TKb dan kaedah lelaran baru (KLB). Keputusan membuktikan bahawa kaedah penghampiran pertama menumpu ke penyelesaian tepat. Kaedah penghampiran kedua menggunakan kaedah Adams Moulton dengan TKb dan KLB manakala kaedah penghampiran ketiga menggunakan kaedah Adam Moulton dengan TKb dan KLB. Dari keputusan berangka, semua kaedah penghampiran kabur yang dicadangkan mengatasi peraturan Trapezium klasik dan kaedah Adam Moulton klasik. Selanjutnya, juga diperhatikan bahawa kaedah-kaedah penghampiran kabur yang dicadangkan lebih tepat berbanding dengan kaedah-kaedah penghampiran kabur sedia ada. Keputusan ini merupakan penambahbaikan penting kepada keputusan sebelumnya bagi masalah-masalah Cauchy. Tambahan pula, sesekat-sesekat kabur telah digunakan dalam dua kaedah penghampiran dengan TKb dan kaedah satu langkah untuk menyelesaikan sistem persamaan-persamaan pembezaan. Kaedah-kaedah penghampiran pertama menggunakan kaedah Euler dengan TKb dan kaedah-kaedah penghampiran kedua menggunakan peraturan Trapezium dengan TKb. Dari keputusan-keputusan berangka, dapat diperhatikan bahawa kedua-dua kaedah penghampiran kabur menghasilkan keputusan yang lebih tepat berbanding kaedah Euler klasik dan peraturan Trapezium klasik. Keputusan ini merupakan satu penambahbaikan penting kepada keputusan sebelumnya untuk menyelesaikan sistem persamaan-persamaan pembezaan. Perbincangan bersambung dalam bahagian akhir dengan teknik-teknik yang lebih maju. Dalam bahagian akhir, sesekat-sesekat kabur sekata teritlak baru telah digunakan dalam tiga kaedah penghampiran berdasarkan TKb untuk menyelesaikan persamaan-persamaan pembezaan. Selaras dengan tiga kaedah penghampiran bagi masalah Cauchy, peraturan Trapezium dan kaedah Adams Moulton diperbaiki menggunakan TKb dan KLB. Dari penyelesaian-penyelesaian berangka, dapat diperhatikan bahawa kaedah penghampiran kabur baru ini menghasilkan penyelesaian yang lebih tepat berbanding peraturan Trapezium dan kaedah Adams Moulton klasik. Oleh itu, kaedah penghampiran kabur baru ini menyediakan teknik alternatif untuk menyelesaikan persamaan pembezaan dengan penyelesaian yang lebih baik.

New Approximation Methods Based on Fuzzy Transform for Solving Ordinary Differential Equations

ABSTRACT

Real world problems in science and engineering are modelled by using differential equations. In many cases, the differential equations cannot be solved analytically so that numerical methods are required. As time goes on, researchers realized that fuzzy approaches are particularly important to numerical solutions of differential equations. One of fuzzy approaches that has been proposed in the literature is the fuzzy transform (FzT). There has been a growing interest in investigating the properties of fuzzy partitions. In this research, new representations of basic functions are proposed. This is achieved by introducing new generalized uniform fuzzy partitions called power of the triangular and raised cosine generalized uniform fuzzy partitions. The main properties of the new generalized uniform fuzzy partitions are proposed. Further, the simpler form of FzT is given alongside with some of its fundamental results. New theorems and lemmas are proposed and proved mathematically. Then, the new generalized uniform fuzzy partitions are used in three approximation methods based on FzT to solve Cauchy problems. The first approximation method used Trapezoidal rule with FzT and new iteration method (NIM). The results proved that the first approximation method converged to the exact solution. The second approximation method used Adams Moulton method with FzT and NIM while the third approximation method used Adams Moulton method with FzT and NIM. From the numerical results, all the proposed fuzzy approximation methods outperform the classical Trapezoidal rule and classical Adams Moulton method. Further, it is also observed that the proposed fuzzy approximation methods are more accurate in comparison with the existing fuzzy approximation methods. This result is an important improvement to the previous results for Cauchy problems. Furthermore, two approximation methods are used in the fuzzy partitions to solve system of differential equations based on FzT and a one step method. The first approximation method used Euler method with FzT and the second approximation method used Trapezoidal rule with FzT. From the numerical results, it is observed that both fuzzy approximation methods yield more accurate results in comparison with the classical Euler method and classical Trapezoidal rule. This result is an important improvement to the previous results for solving system of differential equations. The discussion continued in last part with more advanced techniques. In the last part, the new generalized uniform fuzzy partitions are used in three approximation methods based on FzT to solve system of differential equations. In accordance with the three approximation methods for Cauchy problem, Trapezoidal rule and Adams Moulton method are improved using FzT and NIM. From the numerical results, it is observed that the new fuzzy approximation methods yield more accurate results in comparison with the classical Trapezoidal rule and classical Adams Moulton method. Hence, the new fuzzy approximation methods provide alternative techniques for solving differential equations with better results.

CHAPTER 1

INTRODUCTION

1.1 Research Overview

For many years scientists were depended on crisp (classical) set theory but in real world there are many application problems which cannot be dealt nor described by the crisp set theory. Zadeh (1965) introduced the concept of fuzzy set theory. Therefore, the authors considered fuzzy set theory quickly and used it in their researches, actually it proved its ability in solving many problems, which help scientists to use it in different fields. Fuzzy set theory has been used to establish a new field, i.e. fuzzy approximation, focusing on approximation properties of fuzzy modeling. Fuzzy transform (FzT) is belong to these models.

There are a number of important integral transforms (Fourier, Laplace, wavelet, etc.) are used as powerful methods for solving various problems in ordinary differential equations (ODEs). They are defined by using different kernels. Fuzzy modeling is considered as a modern method with a non-classical background, performed by modern applications of techniques based on fuzzy modeling. In this work, a particular of transformation, namely FzT is considered. This technique has two transforms: direct and inverse FzT. The idea of FzT was developed by Perfilieva and Haldeeva (2001) as a general method for fuzzy modeling.

More specifically, the FzT establishes a correspondence between a set of continuous functions on an interval of real numbers and a set of n-dimensional real vectors (matrices). A fuzzy partition is a finite collection of fuzzy subsets of the universe that determines a discrete kernel and thus a respective transform. Therefore, FzT can be characterized by a kernel that generates a fuzzy partition of the space. In this work, different types of fuzzy partitions are discussed to obtain greater accuracy in the approximate solution of ODEs on the basis of one fuzzy approximation model.

The use of differential equations in mathematical modeling is of great impact and importance in almost all applied sciences, for example mathematical models of series circuits and mechanical systems attached in series can lead to systems of ODEs often applied in chemical, ecological, engineering applications, biological (Alon, 2006). An increased interest of authors has been notices in the field of ODEs as it has different applications. In many cases, the differential equations cannot be solved analytically so that approximation techniques must be used (Griffiths & Higham, 2010). Thus, an application of fuzzy modeling is considered.

FzT has been proposed as a feasibility fuzzy approximation technique with the aim of being applied to unusual application fields such as numerical solution of ODEs. On the other hand, FzT are provided with powerful tools for dealing with typical problems for implementations of fuzzy set. It has been shown that if the original function is replaced by an approximation model, then a certain simplification of complex computations could be achieved (Perfilieva, 2006).

A fuzzy approach to numerical solutions of ODEs has not yet been deeply investigated although some results have been published (Shmilovici & Maimon, 1998). The FzT to numerical solution of ODEs is still regarded as a modern technique with a non-classical background. Further, FzT as a new transform method has been successfully applied into other mathematical problems as well, including signal processing (Perfilieva & Baets, 2010), data analysis (Perfilieva, Novák, & Dvořák, 2008) and the use of higher-order FzT in time series analysis (Novák, Perfilieva, & Pavliska, 2011).

Furthermore, the application of FzT extended to Cauchy problems and two point boundary value problems (BVPs) are published by Alireza, Zahra, and Irina (2017); Khastan, Perfilieva, and Alijani (2016) as well as other numerical methods using this technique. Also, for special issue on FzT: theoretical aspects and advanced applications (Perfilieva, 2016). The emerging of the new research area has opened a new direction in solving ODEs. In this study, FzT method demonstrated to derive approximate solutions for various classes of ODEs.

1.2 Problem Statement

Real world problems in science and engineering are modelled by using differential equations. In many cases, the differential equations cannot be solved analytically so that numerical methods are required. As times goes on, researchers realized that fuzzy approaches to numerical solutions of differential equations are of particular importance. One of the fuzzy approaches that has been proposed in the literature is FzT.

In view of Fig. 1.1, the dashed line is often very difficult for solve problems ODEs. Therefore, through follow the solid line: it may seem a longer path, but it has the advantage of existence solution. In addition, the technique of FzT has been applied for solving ODEs to compare with numerical classical techniques (Perfilieva, 2003; Perfilieva & Haldeeva, 2001; Perfilieva, Meyer, Baets, & Plšková, 2008). Recently, Alireza et al. (2017); Khastan et al. (2016) have applied the FzT method to approximate solution of first order ODEs and two-point BVPs. Chen and Shen (2014) have constructed an algorithm to obtain the approximate solutions of second order initial value problems. Also, this method is proved in literature review as powerful method for advanced applications (Perfilieva, 2016). From this, this method preferred for finding approximate solutions of ODEs. New iteration method (NIM) have proposed by Daftardar-Gejji and Jafari (2006) for solving functional equations. Solutions obtained by NIM are in the form of rapidly converging infinite series which can be effectively approximated by calculating only first few terms. Later, NIM has been proposed for solving delay differential equations (Sukale & Daftardar-Gejji, 2017).

From this discussion, it can be concluded that there are some cases that need to be focused. The cases are as follows.

(i) Knowledge about fuzzy approaches for ODEs is very limited and incomplete in literature. So, there is still a gap on application of FzT method for solving ODEs.

(ii) The topic of ODEs has been growing and have many applications. The exact analytical solutions of ODEs are often difficult and sometimes impossible to obtain. Consequently, FzT is used.

(iii) FzT has a limitation on fuzzy partitions to make approximation more accurate. There has been a growing interest in investigating the properties of fuzzy partitions. However, the problem arises on how one can effectively construct the basic function of fuzzy partitions. Compared with the study on fuzzy partitions, the results on fuzzy partitions are very limited and need to be explored. Further, the best of our knowledge, the study of power of the generalized uniform fuzzy partition are not studied yet. Furthermore, the approximation methods used one, two and three steps with FzT and NIM for solving generalized Cauchy problem and generalized coupled system of first order ODEs are not studied yet. Finally, the generalized coupled system of first order ODEs with initial value problem using FzT and its theoretical properties are not available in literature.

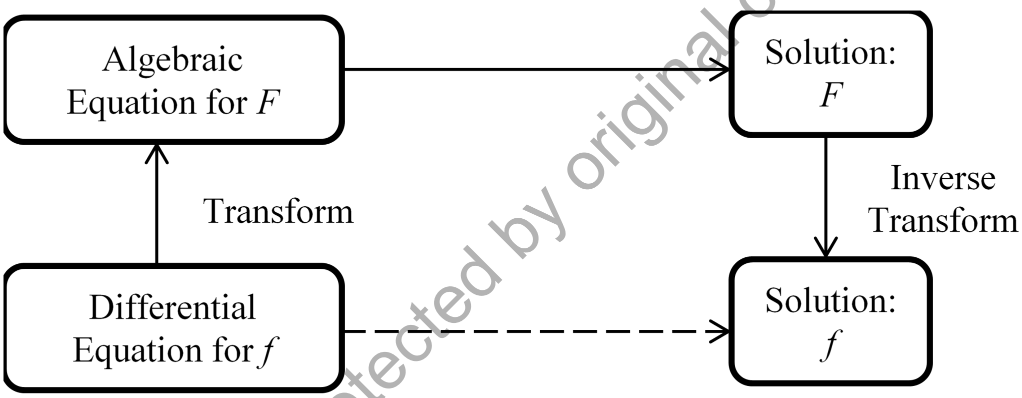

Figure 1.1: Steps Direct and Inverse Transform.

1.3 Research Objectives

The objectives of the research are as follows:

(i) To study new generalized uniform fuzzy partitions of real line, formally called power of the generalized uniform fuzzy partition, and its fundamental results.

(ii) To construct procedures for solving generalized Cauchy problems using the three conventional numerical methods: Trapezoidal rule (one step) and Adams Moulton method (two and three step modifications), NIM based on the FzT. Also, a comparison of numerical results with existing results.

(iii) To extend procedures for solving generalized coupled system of first-order of ODEs using the two conventional numerical methods: Euler method (one stage) and Trapezoidal rule (two stage) based on the FzT and comparison of the results with conventional numerical methods.

(iv) To compare and derive three procedures for solving generalized coupled system of first-order of ODEs using the three conventional numerical methods: Trapezoidal rule (one step) and Adams Moulton method (two and three step modifications), NIM based on the FzT.

1.4 Scope and Limitation of Research

The goal of this study is to bridge standard mathematical methods and methods for the construction of fuzzy approximation models. Consequently, the research focused to extend the applicability FzT, specifically, fuzzy partition. Some fundamental results of FzT are also proposed and proved. From there, the solving procedures are developed in this thesis. Applications on the class of ODEs, including Cauchy problem and system of ODEs are also done afterwards. The classes of ODEs considered in this thesis are presented with respect to the power of the generalized uniform fuzzy partition concept. Further, the convergence analysis of some procedures is theoretically justified and illustrated on various examples.

1.5 Thesis Outline

The thesis is organized as follows. Main definitions, claims and numerical methods accepted FzT are recalled in Chapter 2. The literature reviews on the development of FzT are also presented. This would help to understand the rest of this thesis. Later in Chapter 3, FzT is introduced and some properties and theorems are proposed alongside. Later in Chapter 4, procedures for solving Cauchy problems are provided, together with an example demonstration. While in Chapter 5, the procedures for solving system of ODEs are extended. These procedures are demonstrated on numerical examples. In Chapter 6, the procedures for finding the solution of system of ODEs are constructed, where these procedures are applied on numerical examples afterwards to show the ability of the method. Finally, in Chapter 7, conclusion is drawn and the research contributions are highlighted.

1.6 Summary

In this chapter, an overview of this research has been provided. Some backgrounds and motivations of this research are given. Next, the problems are identified and highlighted. The objectives of this research are proposed. Scope and limitation of this research are stressed. Finally, the outline of this thesis is provided in details.

CHAPTER 2

THEORETICAL BACKGROUND AND LITERATURE REVIEW

2.1 Introduction

The main purpose of this chapter is to provide a brief the introduction to basic concepts. These concepts will be used throughout the thesis as a background for basic definitions, necessary notations and theoretical background. Also, this chapter focuses on proposing the literature reviews on the development of FzT. Fuzzy modeling, fuzzy partitions, applications of FzT, numerical methods for ODEs are presented.

2.2 Fuzzy Sets

Fuzzy sets are generalizations of the classical sets represented by their characteristic (membership) functions. Membership functions were introduced by Zadeh (1965). A fuzzy number $u : R \to [0, 1]$ is a fuzzy subset of the real line, satisfying the following properties : u is normal, u is a convex fuzzy set, u is upper semi continuous on R and $cl\,\{s \in R \mid u(s) > 0\}$ is compact where cl denotes the closure of a subset.

2.3 Fuzzy Partitions

Throughout this section, the interval $[a, b]$ on R as a universe is considered. Fuzzy sets on $[a, b]$ will be identified with their membership functions mapping from $[a, b]$ into $[0, 1]$. Assume that the interval $[a, b]$ as a real domain. In this section, definitions and claims remind that were introduced and proved by Holčapek, Perfilieva, Novák, and Kreinovich (2015); Perfilieva (2006, 2015); Perfilieva, Daňková, and Bede (2011); Stefanini (2011). The various types of fuzzy partitions with their special properties are summarized. Later in Chapter 3, new representations of uniform fuzzy partitions are introduced and some properties and theorems are proposed.

2.3.1 Fuzzy Partition with the Ruspini Condition

The fuzzy partition of FzT with orthogonality axiom was presented by Perfilieva and Haldeeva (2001) and was extensively investigated by Perfilieva (2006). The following definition is taken from Perfilieva (2006):

Definition 2.1. *Let* $\{x_1, \cdots, x_n\} \in [a,b]$ *and set* $x_1 = a,\ x_n = b$ *and* $n \geq 2$. $A_1, \ldots, A_n$ *are the fuzzy sets, called basic functions. The fuzzy sets* $A_1, \ldots, A_n \subset [a,b]$ *are fuzzy partitions of* $[a,\, b]$ *if they fulfill the following conditions for* $k = 1, \ldots, n$*: (set* $x_0 = a$ *and* $x_{n+1} = b$ *for uniformity of notation)*

(i) (Continuity, Normality, Positivity and locality)

$A_k(x) : [a,\, b] \to [0,1]$ *is continuous with* $A_k(x_k) = 1$, $A_k(x) > 0$ *if* $x \in (x_{k-1}, x_{k+1})$ *and* $A_k(x) = 0$ *if* $x \notin (x_{k-1}, x_{k+1})$*;*

(ii) Monotonicity (Convexity)

The fuzzy sets $A_k(x)$ *strictly increases on* $[x_{k-1}, x_k]$, $k = 2, \ldots, n$, *and the fuzzy sets* $A_k(x)$ *strictly decreases on* $[x_k, x_{k+1}]$, $k = 1, \ldots, n-1$*;*

(iii) (Orthogonality) The Ruspini condition is given that

$$\forall x \in [a, b], \sum_{k=1}^{n} A_k(x) = 1. \tag{2.1}$$

The fuzzy partitions of $[a, b]$ are h-uniform if its nodes $x_1, \ldots, x_n$, where $n \geq 2$, are equidistant. This means that $x_k = a + h(k-1)$, $k = 1, \ldots, n$, where $h = \frac{b-a}{n-1}$, $n \geq 2$, and the two additional properties are fulfilled:

(i) $A_k(x_k - x) = A_k(x_k + x)$, $k = 2, \ldots, n-1$, for all $x \in [0, h]$ and

(ii) $A_k(x) = A_{k-1}(x-h)$ and $A_{k+1}(x) = A_k(x-h)$, $k = 2, \ldots, n-1$, for all $x \in [x_k, x_{k+1}]$.

The following formulas represent generic fuzzy partitions with the Ruspini condition and triangular functions for $k = 2, \ldots, n-1$ and $h_k = x_{k+1} - x_k$:

$$\left.\begin{array}{l}
A_1(x) = \begin{cases} 1 - \frac{(x-x_1)}{h_1}, & x \in [x_1, x_2], \\ 0 & \text{otherwise}, \end{cases} \\[2em]
A_n(x) = \begin{cases} \frac{(x-x_{n-1})}{h_{n-1}}, & x \in [x_{n-1}, x_n], \\ 0, & \text{otherwise}, \end{cases} \\[2em]
A_k(x) = \begin{cases} \frac{(x-x_{k-1})}{h_{k-1}}, & x \in [x_{k-1}, x_k], \\ 1 - \frac{(x-x_k)}{h_k}, & x \in [x_k, x_{k+1}], \\ 0, & \text{otherwise}, \end{cases}
\end{array}\right\} \tag{2.2}$$

To make this fuzzy partition h-uniform, it is sufficient to put $h_k = h$, for all k. The