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Irreversibility in an Optical Parametric Driven Optomechanical System

Obinna Abah, Collins O. Edet, Norshamsuri Ali, Berihu Teklu, and Muhammad Asjad**

This study investigates the role of nonlinearity via optical parametric oscillator on the entropy production rate and quantum correlations in a hybrid optomechanical system. Specifically, the modified entropy production rate of an optical parametric oscillator placed in the optomechanical cavity is derived, which is well described by the two-mode Gaussian state. The irreversibility and quantum mutual information associated with the driving the system far from equilibrium are found to be controlled by the phase and strength of nonlinearity. This analysis shows that the system entropy flow, heating, or cooling, are determined by choosing the appropriate phase of the self-induced nonlinearity. It is further demonstrated that this effect persists for a reasonable range of cavity decay rate.

1. Introduction

Hybrid quantum systems exploit different physical components with complementary functionalities for efficient multitasking tasks.[1] They catalyze novel fundamental research in quantum physics, condensed matter physics, and mesoscopic physics by providing platforms to investigate various phenomena at the quantum regimes. However, hybrid cavity optomechanical systems are attracting a lot of attention due to their integration versatility, promising reliable quantum controllability, and long coherent time.[2] These systems are also playing a prominent role in realizing the broad range of novel applications in quantum technologies, including quantum metrology,[3] quantum communication,[4] as quantum transducers,[5,6] for fundamental tests of quantum mechanics[7–9] or realizing quantum thermal machines.[10–12]

The performances of thermal heat machines were successfully analyzed within the established framework of classical thermodynamics[13] and played a prominent role during the industrial revolution. In the last decades, thermodynamics has been extended to classical small devices/systems operating far from equilibrium by considering the fluctuations via stochastic thermodynamics.[14,15] In view of harnessing the promises of quantum technologies, there has been a tremendous interest in the thermodynamical analysis of devices operating in quantum regime.[16–20] In addition, with the advancement in fabrication technology, various experiments studying nonequilibrium thermodynamics in the quantum regime have been realized.[21–23] Recently, a nonlocal thermoelectric heat engine has been proposed with hybrid superconducting devices, allowing for the coexistence of various thermal operations.[24] The feasibility of a quantum optomechanical Stirling heat engine, powered by feedback control, has been theoretically studied within the confines of the current experimental framework.[25]

The presence of the degenerate optical parametric oscillator (OPO) inside a dissipative cavity optomechanical induces a nonlinear interaction due to a second-order nonlinearity of optical crystals. These systems have been proposed to considerably enhance the entanglement,[26] mechanical squeezing,[27,28] the cooling of the micromechanical mirror,[29–31] force sensing of the system,[32] and improve the precision of position detection.[33] The phase-sensitive amplifier offers a promising applications in quantum communication[34,35] and quantum sensing.[36] On the otherhand, nonlinearity has been shown to be a useful resource for generating non-classical quantum states.[37–39] Nevertheless, the impact of nonlinearity on the non-equilibrium thermodynamic characteristics of hybrid optomechanical systems remains to be elucidated.

In recent time, a formulation for the characterization of irreversible entropy production of quantum systems interacting with nonequilibrium reservoirs that combines quantum

O. Abah
School of Mathematics, Statistics, and Physics
Newcastle University
Newcastle upon Tyne NE1 7RU, UK
E-mail: obinna.abah@newcastle.ac.uk

C. O. Edet
Institute of Engineering Mathematics
Universiti Malaysia Perlis
Arau, Perlis 02600, Malaysia

N. Ali
Advanced Communication Engineering (ACE) Centre of Excellence
Universiti Malaysia Perlis
Kangar, Perlis 01000, Malaysia

B. Teklu, M. Asjad
Department of Mathematics
Khalifa University
Abu Dhabi 127788, United Arab Emirates
E-mail: muhammad.asjad@unicam.it

B. Teklu
Center for Cyber-Physical Systems (C2PS)
Khalifa University
Abu Dhabi 127788, United Arab Emirates

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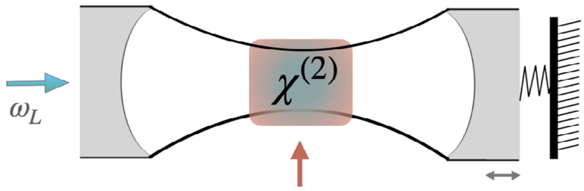

Figure 1. A schematic description of the optomechanical system incorporating intracavity squeezing generated by the driven nonlinear crystal.

phase-space methods and the Fokker-Planck equation, has been put forward.[40–43] The irreversible entropy production in mesoscopic quantum systems has been experimentally measured in two different driven-dissipative quantum systems realized by coupling bosonic systems to high-finesse cavities.[44] Shahidani and Rafiee have studied the role of self-correlation on irreversible thermodynamics in a parametrically driven-dissipative system.[45] It is established that optimizing thermal device efficiency is closely linked to the reduction of nonequilibrium entropy production.[46,47]

In this paper, we present the irreversibility generated in the stationary state of a nonlinear crystal OPO placed inside the optomechanical cavity. We demonstrate that the entropy production rate and the corresponding quantum correlations of the cavity optomechanical setup are modified by this self-induced nonlinearity. We also show that the irreversibility in the system enhances by the squeezing generated from the nonlinear medium. The direction of entropy flow in the quantum system is controlled by the squeezed cavity phase. In addition, we analyze the role of the self-induced nonlinearity on the quantum correlations of the optomechanical system.

The rest of the paper is structured as follows. In Section 2 we describe the full theoretical model Hamiltonian of the hybrid optomechanical setup. Following the Heisenberg-Langevin procedure, we linearize the system dynamics, where we focus explicitly on the Gaussian states. Section 3 presents the results and discussions of the entropy production rate and the quantum correlations of the model Hamiltonian. First, in Section 3.1 we present the analysis of the entropy production rate of a nonlinear hybrid optomechanical system while the Section 3.2 details the behavior of its quantum mutual information and quantum discord. Finally, we conclude in Section 4.

2. Model

We consider an optomechanical system, which consists of a Fabry-Perot cavity with a moving end mirror driven by a laser field of frequency ω_L through its fixed mirror, as shown in **Figure 1**. The optical cavity (with resonance frequency ω_C) contains a nonlinear crystal of nonlinearity $\chi^{(2)}$ that is pumped with classical driving frequency $2\omega_L$ and phase θ. The mechanical resonator undergoes oscillation with frequency ω_b, which modulates the cavity resonance frequencies. The Hamiltonian of the system in a rotating frame at the frequency ω_L of the pump field reads ($\hbar = 1$)[48,49]

$$H = \Delta_0 \hat{a}^\dagger \hat{a} - i\left(\eta^* \hat{a} - \eta \hat{a}^\dagger\right) + \xi\left(e^{i\theta}\, \hat{a}^{\dagger 2} + e^{-i\theta} \hat{a}^2\right) + \omega_b \hat{b}^\dagger \hat{b} + g \hat{a}^\dagger \hat{a}(\hat{b} + \hat{b}^\dagger) \tag{1}$$

where $\Delta_0 = \omega_C - \omega_L$ is the cavity detuning with ω_C is the cavity frequency, and $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operators of the cavity mode. The mechanical mode annihilation and creation operators are denoted by $\hat{b}$ and $\hat{b}^\dagger$. The first and fourth terms describe the free energy of the cavity mode and mechanical mode, respectively. The second term in Equation (1) describes the driving of the cavity with the laser rate $|\eta| = \sqrt{2\kappa\mathcal{R}/\omega_L}$ where $\mathcal{R}$ is the laser power and κ is the cavity decay rate. The third term is the parametric amplification of the optical cavity where ξ is the strength of nonlinear interaction, which is proportional to the amplitude of the classical pump and second order nonlinear susceptibility, and θ is the phase of the parametric amplification.[50] The fifth term of Equation (1) represents the radiation-pressure interaction between the cavity and the mechanical mode modes with single photon optomechanical-coupling strength $g = \sqrt{1/2M\omega_b}\,\omega_C/L$,[49,51] where L is the cavity length in the absence of the cavity field and M is the mass of the mechanical resonator.

Here, we remark that nonlinearity induces squeezing in the cavity mode. It can be used to generate a strong mechanical squeezing in an optomechanical system.[49] The nonlinearity could also be employed to enhance optomechanical cooling,[30] which has been demonstrated recently in magnetomechanical setup.[52]

In what follows, we apply the input-output formalizm to account for the dissipation and fluctuations introduced by the coupling of the system with an environment.[48,53] Following the Heisenberg-Langevin approach,[53] the system dynamics including the dissipation caused by system-environment couplings and their corresponding noises, the quantum Langevin equation for the optomechanical system are

$$\begin{aligned}\dot{\hat{a}} &= -\left(\kappa + i\Delta_0\right)\hat{a} + \eta - 2i\xi e^{i\theta}\, \hat{a}^\dagger - ig\left(\hat{b} + \hat{b}^\dagger\right)\hat{a} + \sqrt{2\kappa}\hat{a}_{\rm in} \\ \dot{\hat{b}} &= -(\gamma + i\omega_b)\hat{b} - ig\hat{a}^\dagger\hat{a} + \sqrt{2\gamma}\hat{b}_{\rm in}\end{aligned} \tag{2}$$

where γ is the damping rate of mechanical resonator, $\hat{a}_{in}$ is the *zero* mean, i.e., $\langle\hat{a}_{in}\rangle = 0$, input noise operator for optical mode with only non-zero correlation $\langle\hat{a}_{in}(t)\hat{a}^\dagger_{in}(t')\rangle = \delta(t - t')$, while $\hat{b}_{in}$ is the input noise operator with zero mean, $\langle\hat{b}_{in}\rangle = 0$, associated with the mechanical oscillator and described by the correlation function $\langle\hat{b}_{in}(t)\hat{b}^\dagger_{in}(t')\rangle = (n_b + 1)\delta(t - t')$ with $n_b = \{e^{(\hbar\omega_b/K_B T)} - 1\}^{-1}$ is the mean thermal occupation number of the mechanical mode at the temperature T.

To linearize the nonlinear set of equations, Equation (2), we can treat the Heisenberg operators as the sum of their mean values and small quantum fluctuations, i.e, $\hat{a} = a_s + \delta\hat{a}$ and $\hat{b} = b_s + \delta\hat{b}$, where $\delta\hat{a}$ and $\delta\hat{b}$ are small quantum fluctuations around the steady state fields a_s and b_s. Inserting these expressions into the Langevin equations of Equation (2), these latter decouple into a set of nonlinear algebraic equations for the steady state values and a set of quantum Langevin equations for the fluctuation operators.[54] The steady-state values are

$$a_s = \frac{\eta(\kappa - i\Delta_a) - 2i\xi\eta^* e^{i\theta}}{\kappa^2 + \Delta_a^2 - 4\xi^2}, \quad b_s = -i\frac{g|a_s|^2}{\gamma + i\omega_b} \tag{3}$$

where $\Delta_a = \Delta_0 + 2g\mathrm{Re}b_s$ is the effective detuning and the phase of input driving laser is choosen such that a_s is *real*. The corresponding linearized quantum Langevin equations for the quantum fluctuation operators dynamics are given by

$$\delta\dot{\hat{a}} = -(\kappa + i\Delta_a)\delta\hat{a} - iG\left(\delta\hat{b} + \delta\hat{b}^\dagger\right) + \chi\delta\hat{a}^\dagger + \sqrt{2\kappa}\delta\hat{a}_{\mathrm{in}}$$

$$\delta\dot{\hat{b}} = -(\gamma + i\omega_b)\delta\hat{b} - iG(\delta\hat{a} + \hat{a}^\dagger) + \sqrt{2\gamma}\hat{b}_{\mathrm{in}} \tag{4}$$

where $G = 2ga_s$ is the effective optomechanical coupling strength, and $\chi = -2i\xi e^{i\theta}$ represents the effective nonlinear interaction, with the amplitude $|\chi|$ and $\phi = \tan^{-1}[\mathrm{Im}\chi/\mathrm{Re}\chi]$.

Let now introduce the optical cavity mode quadratures $\delta\hat{x}_a = (\delta\hat{a} + \delta\hat{a}^\dagger)/\sqrt{2}$ and $\delta\hat{p}_a = (\delta\hat{a} - \delta\hat{a}^\dagger)/i\sqrt{2}$. Similarly, the quadratures of the mechanical mode are $\delta\hat{x}_b = (\delta\hat{b} + \delta\hat{b}^\dagger)/\sqrt{2}$ and $\delta\hat{p}_b = (\delta\hat{b} - \delta\hat{b}^\dagger)/i\sqrt{2}$. Likewise, the corresponding Hermitian input noise operators $\hat{x}_{j,\mathrm{in}} = (\hat{j}_{\mathrm{in}} + \hat{j}^\dagger_{\mathrm{in}})/\sqrt{2}$ and $\hat{p}_{j,\mathrm{in}} = (\hat{j}_{\mathrm{in}} - \hat{j}^\dagger_{\mathrm{in}})/i\sqrt{2}$ ($j = a, b$). Then, the quantum Langevin equations for the quadratures can be written in the compact matrix form

$$\dot{\hat{R}}(t) = A\,\hat{R}(t) + \hat{R}_{\mathrm{in}}(t) \tag{5}$$

where the quadratures vector $\hat{R}(t) = (\delta\hat{x}_a, \delta\hat{p}_a, \delta\hat{x}_b, \delta\hat{p}_b)^T$ and the noises vector $\hat{R}_{\mathrm{in}} = (\sqrt{2\kappa}\,\hat{x}_{a,\mathrm{in}}, \sqrt{2\kappa}\,\hat{p}_{a,\mathrm{in}}, \sqrt{2\gamma}\,\hat{x}_{b,\mathrm{in}}, \sqrt{2\gamma}\,\hat{p}_{b,\mathrm{in}})^T$. The corresponding drift matrix A can be explicitly determined and depends on the set of parameters characterizing the dynamics of the two-mode system; it reads

$$A = \begin{pmatrix} -\kappa + |\chi|\cos(\phi) & \Delta_a + |\chi|\sin(\phi) & 0 & 0 \\ -\Delta_a + |\chi|\sin(\phi) & -\kappa - |\chi|\cos(\phi) & G & 0 \\ 0 & 0 & -\gamma & \omega_b \\ G & 0 & -\omega_b & -\gamma \end{pmatrix} \tag{6}$$

The system dynamical equations should be stable in order for a steady state to exist and the stability condition for system can be formalized in terms of the Routh-Hurwitz criterion,[55] which we employed in our characterization of the dynamics. This is achieved if the real part of the spectrum of the drift matrix A is negative, i.e., all the eigenvalues of the drift matrix A have negative real parts.

3. Results and Discussion

We now proceed to analyze the irreversible entropy production rate and the quantum correlation profiles of our setup consisting of an optical parametric oscillator inside the dissipative-driven optomechanical cavity. We will focus on the how these physical quantities are affected by the presence of the OPO when the system reaches its stationary state.

3.1. Entropy Production and Correlations Matrix

Here, we follow the framework that characterizes the entropy production as the correlation between a system and a reservoir.[40,56] Due to the linearized dynamics of the quantum fluctuations and since all the quantum noise terms are Gaussian, the resulting steady state of the system is a continuous-variable Gaussian state which can be fully characterized by the 4×4 stationary correlation matrix (CM) $\mathcal{V}$, with components $\mathcal{V}_{ij} = \langle\delta\hat{R}_i(\infty)\delta\hat{R}_j(\infty) + \delta\hat{R}_j(\infty)\delta\hat{R}_i(\infty)\rangle/2$. The elements of the quantum CM must satisfy the uncertainty relation $\mathcal{V} + i\Omega \geq 0$, where Ω_{ij} are the elements of the symplectic matrix given by the Heisenberg uncertainty principle ($[\hat{R}_i, \hat{R}_j] = i\Omega_{ij}$).[57] We assume that the first moments are null, which can be achieved by choosing a suitable displacement in the phase space. The equation of motion for the covariance matrix as $\dot{\mathcal{V}} = A\mathcal{V} + \mathcal{V}A^T + D$, where $D = \mathrm{diag}\{\kappa, \kappa, \gamma(2n_b + 1), \gamma(2n_b + 1)\}$ is the diffusion matrix. Considering that the two reservoirs are prepared at different temperatures, this leads to the breaking of the detailed balance and takes the system to nonequilibrium state. When the system is stable, the Lyapunov equation for the nonequilibrium steady state covariance matrix, $\lim_{t\to\infty}\mathcal{V}(t) = \mathcal{V}^s$, reads

$$A\mathcal{V}^s + \mathcal{V}^s A^T = -D \tag{7}$$

Considering the Gaussian bosonic nature of the system, this allows us to make a connection between the Wigner entropy of the system and the covariance matrix ($\mathcal{V}$) as:

$$\mathcal{W}(u) = \frac{1}{(2\pi)^n\sqrt{\det\mathcal{V}}}e^{-\frac{1}{2}u^T\mathcal{V}^{-1}u} \tag{8}$$

where the Wigner function ia a quasi-probability distribution in phase space (n is the number of the bosonic modes) and always positive. It has been established that the entropy S of a non-equilibrium quantum system can be evaluated using the Shannon entropy of the Wigner function,[41] and reads

$$S = -\int du\,\mathcal{W}(u)\log\mathcal{W}(u) \tag{9}$$

This Wigner entropy follows density matrix description and has been shown to be suitable quantifier for the irreversibility of quantum Gaussian platforms.[44,45]

Consequently, the open dynamics of the optomechanical system can be described in terms of Fokker-Planck equations based on the Wigner function of the joint system. Following the approach recently put forward, the steady-state entropy production rate Π_s is given by refs. [40, 44],

$$\begin{aligned}\Pi_s &= 2\mathrm{Tr}\left((A^{\mathrm{irr}})^T D^{-1} A^{\mathrm{irr}}\,\mathcal{V}^s\right) + \mathrm{Tr}\left(A^{\mathrm{irr}}\right) \\ &= 2\kappa\left(\mathcal{V}^s_{11} + \mathcal{V}^s_{22} - 1\right) + 2\gamma\left(\frac{\mathcal{V}^s_{33} + \mathcal{V}^s_{44}}{2n_b + 1} - 1\right) \\ &= \mu_a + \mu_b,\end{aligned} \tag{10}$$

where $A^{\mathrm{irr}} = \mathrm{diag}\{-\kappa, -\kappa, -\gamma, -\gamma\}$ and μ_a (μ_b) corresponds to the contributions to Π_s from the cavity (mechanical) mode, respectively. When the system is in the equilibrium state, we have $\mathcal{V}^s_{11} + \mathcal{V}^s_{22} = 1$, $\mathcal{V}^s_{33} + \mathcal{V}^s_{44} = 2n_b + 1$, and hence, $\Pi_s = 0$. From the Lyapunov equation, Equation (7), the diagonal and off-diagonal terms of the covariance matrix are related as follows;

$$\mathcal{V}^s_{11} = \frac{\kappa}{2}\frac{1}{\kappa - |\chi|\cos(\phi)} + \frac{\Delta_a + |\chi|\sin(\phi)}{\kappa - |\chi|\cos(\phi)}\mathcal{V}^s_{12}$$

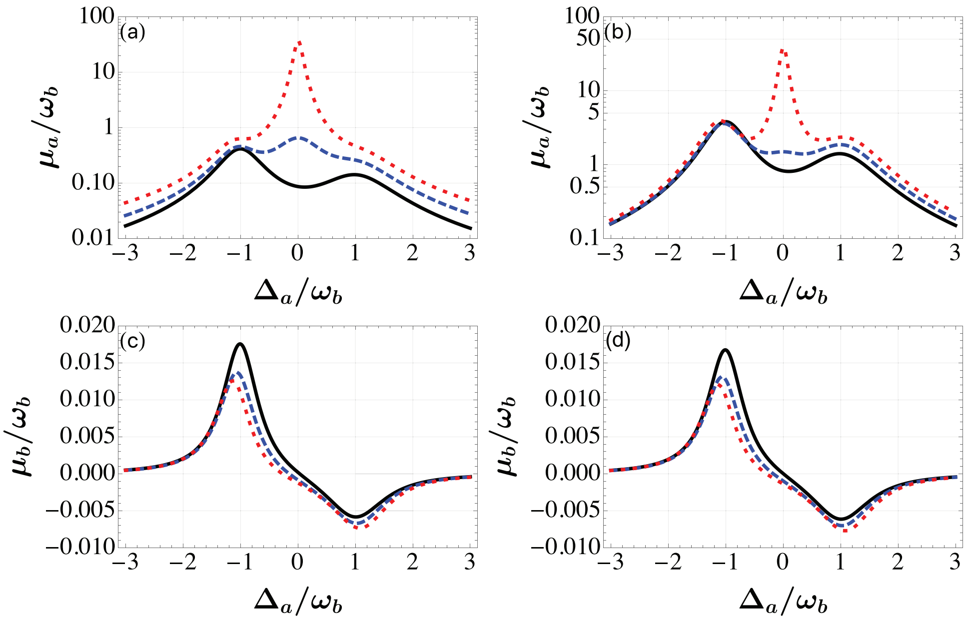

Figure 2. Scaled entropy production contributions μ_a/ω_b and μ_b/ω_b against the normalized detuning Δ_a/ω_b for different strength of the nonlinear self-interaction of the mode a. The black solid curves corresponds to $|\chi|=0$, the dashed blue curves corresponds to $|\chi|=0.3\,\omega_b$ and the dotted red curves corresponds to $|\chi|=0.49\,\omega_b$. a,c) represent the plots when the number of thermal excitation $n_b=10$ while the b,d) denotes the case of $n_b=100$. The other parameters are $\gamma=10^{-2}\omega_b$, $\kappa=0.5\,\omega_b$, $\phi=0.8\pi$ and $G=0.1\,\omega_b$.

$$\mathcal{V}^s_{22}=\frac{\kappa}{2}\frac{1}{\kappa+|\chi|\cos(\phi)}+\frac{G}{\kappa+|\chi|\cos(\phi)}\mathcal{V}^s_{23}-\frac{\Delta_a-|\chi|\sin(\phi)}{\kappa+|\chi|\cos(\phi)}\mathcal{V}^s_{12}$$

$$\mathcal{V}^s_{33}=\frac{2n_b+1}{2}+\frac{\omega_b}{\gamma}\mathcal{V}^s_{34}$$

$$\mathcal{V}^s_{44}=\frac{2n_b+1}{2}+\frac{G}{\gamma}\mathcal{V}^s_{14}-\frac{\omega_b}{\gamma}\mathcal{V}^s_{34} \tag{11}$$

Hence, the entropy production rate can be expressed using off-diagonal elements of the covariance matrix as

$$\Pi_s=\frac{2\kappa|\chi|^2\cos^2(\phi)}{\kappa^2-|\chi|^2\cos^2(\phi)}+\frac{4\kappa|\chi|\cos(\phi)\left[\Delta_a+\kappa\tan(\phi)\right]}{\kappa^2-|\chi|^2\cos^2(\phi)}\mathcal{V}^s_{12}+\frac{2G}{2n_b+1}\mathcal{V}^s_{14}+\frac{2\kappa G}{\kappa+|\chi|\cos(\phi)}\mathcal{V}^s_{23} \tag{12}$$

Equation (12) encapsulates the complete information regarding the impact of the OPO on the irreversibility of a driven-dissipative optomechanical system. It shows that in the absence of crystal nonlinearity ($|\chi|=0$), we recover the results presented in ref. [40]. From Equation (12), even for a small vanishing coupling ($G=0$), Π_s remains non-zero and explicitly depends on the contribution of the optical cavity mode. This is attributed to the nonlinear interaction driving the optical mode of the system into a nonequilibrium state, an effect absent when $|\chi|=0$. Consequently, for finite nonlinear interaction values $|\chi|$ and ϕ, Π_s is modified due to the combined contributions from both the cavity and mechanical mode dynamical variables.

To numerically illustrate the influence of nonlinear contributions on the entropy production of the OPO cavity optomechanical system, we consider the resolved sideband regime where $\kappa<\omega_b$. In addition, we consider the following system parameter range: the cavity frequency $\omega_C\approx 2\pi\times 4.93$ GHz, cavity decay rate $\kappa\approx 2\pi\times 215$ KHz, mechanical resonator frequency $\omega_b\approx 2\pi\times 65$ MHz, corresponding mechanical damping rate $\gamma\approx 2\pi\times 15$ KHz, and single-photon optomechanical coupling strength $g\approx 2\pi\times 10$ Hz, which are realistic with the current optomechanical setup experiments.[58–61] In **Figure 2**, we present the individual contributions μ_i ($i=a,b$) to the entropy production rate as a function of normalized detuning for different initial thermal average occupations of the mechanical mode. Specifically, in Figure 2a–d, we plot the rescaled cavity [mechanical] mode contribution to the total entropy production rate as a function of normalized detuning Δ_a/ω_b for different values of $|\chi|$. In the limit of large detuning $\Delta_a\gg\omega_b$, the two modes become effectively decoupled, resulting in a vanishing Π_s. For the cavity component μ_a, the behavior appears symmetric between the red-detuned and blue-detuned regimes, with amplification peaking at $\Delta_a=0$. From Figure 2a,b we observe that the increased entropy production rate (irreversibility) associated with the presence of crystal nonlinearity $|\chi|\neq 0$ tends to vanish at $\Delta_a=\omega_b$ and $\Delta_a=-\omega_b$. In comparison to the μ_b, Figure 2c,d, there is no appreciable effect on the behavior of the entropy production rate outside the peaks for both changes in $|\chi|\neq 0$ and the number of thermal excitations n_b.

The contribution of the cavity mode to the entropy production rate, denoted as μ_a, is always positive and increases as the nonlinear interaction $|\chi|$ becomes larger. On the other hand, Figure 2c,d reveal sign changes in the mechanical mode component, μ_b, although this is reasonable because the sum of the two components

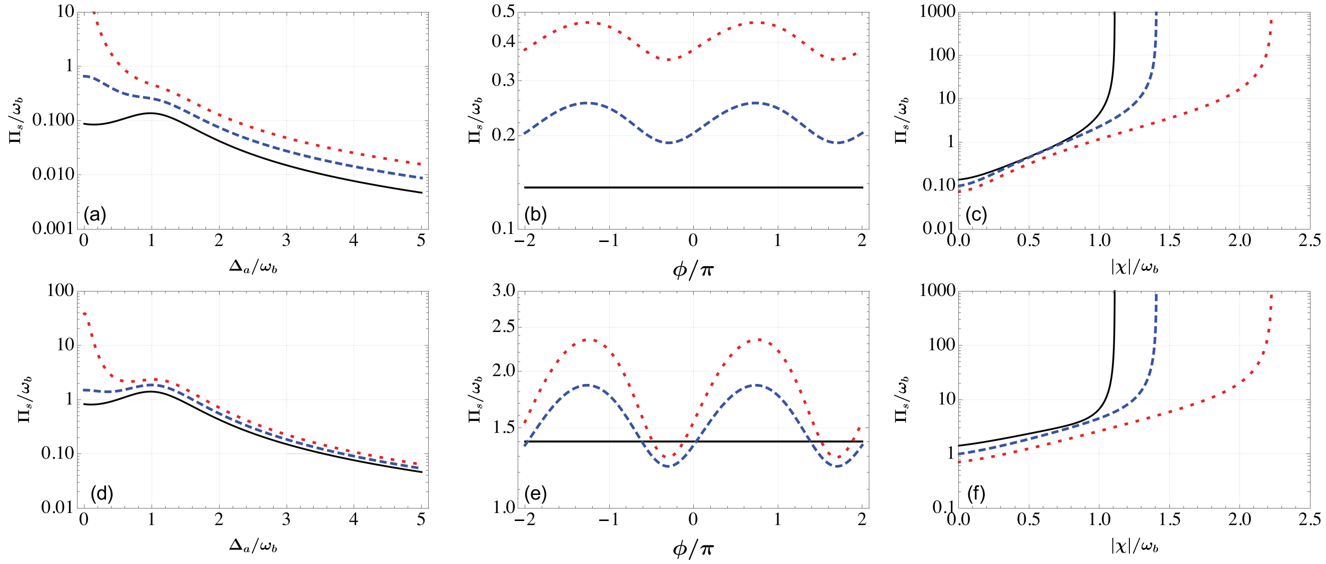

Figure 3. Entropy production rate Π_s as function of normalized detuning Δ_a/ω_b (first column) and phase ϕ/π (second column) for different values of $|\chi|$. The solid black curve corresponds to $|\chi|=0$, while the dashed blue curve shows the case of $|\chi|=0.5\,\omega_b$, and dotted red curve is for $|\chi|=\omega_b$, taking $\phi=0.8\,\pi$, $\gamma=10^{-2}\,\omega_b$, $\kappa=0.5\,\omega_b$, and $G=0.1\,\omega_b$. c) The third column, (c,f)), plot of Π_s as function of strength of nonlinearity $|\chi|/\omega_b$ for different values of cavity decay rates $\kappa=0.5\omega_b$ (black curve), $\kappa=\omega_b$ (blue curve) and $\kappa=2\omega_b$ (red curve) for $\Delta_a=\omega_b$. The other parameters are $\gamma=10^{-2}\omega_b$, $\phi=0.8\,\pi$ and $G=0.1\,\omega_b$. a–c) [the first row] represent the plots of thermal excitation $n_b=10$ while d,f) [second row] are the plots for $n_b=100$.

is non-zero (i.e., $\mu_a+\mu_b>0$). The behavior of μ_b in the effective red-detuned region, $\Delta_a>0$ ($\Delta_a<0$), captures the optomechanical cooling (heating) signature. This means that increasing the nonlinearity $|\chi|$ enhances the entropy flow from the mechanical resonator to the optical cavity, thereby lowering the effective temperature of the resonator.[62] Further quantitative understanding can be obtained when considering the small coupling limit, i.e., $G/\omega_b \ll 1$. In this case, we can expand the cavity (μ_a) and the mechanical (μ_b) components in a power series of G, resulting in the expressions $\mu_a=2\kappa|\chi|^2/(\Delta_a^2+\kappa^2-|\chi|^2)+\mathcal{O}(G^2)$ and $\mu_b=4n_b\gamma+\mathcal{O}(G^2)$. This shows that as Δ_a approaches zero, the nonlinearity $|\chi|$ influences μ_a, whereas μ_b remains unaffected. In addition, both contributions μ_a and μ_b are peaked at the two sidebands due to the hybridization of the two modes.

In **Figure 3**, we first present the entropy production rate Π_s against the rescaled detuning Δ_a/ω_b and the phase ϕ for different values of the nonlinear interaction parameter $|\chi|$. Second, we present Π_s against rescaled $|\chi|$ for different cavity decay rates κ. For the stable parameter range of the system, an increase in the nonlinearity contribution results in an enhancement of the entropy production rate. In Figure 3a–c, we consider a low number of thermal excitations for the mechanical oscillator ($n_b=10$), while in Figure 3d–f, we assume a much higher initial occupation number ($n_b=100$). Focusing on the red-detuned parameter regime, Figure 3a shows that the entropy production rate Π_s increases with the increasing strength of crystal nonlinearity $|\chi|$, and it diverges towards resonance ($\Delta_a=0$). From Figure 3d, the case of high number of thermal excitations, it shows that the amount of irreversibility decreases but still finite even when $\Delta_a \gg \omega_b$. Thus, the initial thermal excitation number of the mechanical mode influences the Π_s profile.

To explore the impact of the squeezing phase on Π_s, in Figure 3b,e, we show Π_s as a function of the phase ϕ/π for different values of $|\chi|$. Specifically, Figure 3b,e are calculated for the number of excitations $n_b=10$ and $n_b=100$, respectively. We observe an oscillating behavior of irreversibility with the variation of the nonlinear interaction phase. The maximum (minimum) Π_s occurs at $\phi\approx 0.8n\pi$ or $-1.25n\pi$ ($\phi\approx 1.7n\pi$ or $-0.25n\pi$), where n is an integer. For a large initial number of thermal excitations ($n_b=100$) and $|\chi|\neq 0$, the minimum Π_s is lower than the case of $|\chi|=0$ (see Figure 3e). This clearly shows that the induced entropy flow associated with driving the system into a nonequilibrium state depends on the chosen nonlinear interaction phase ϕ. This means that the nonlinear interaction medium can be used to control the direction of heat flow in a quantum system, which could assist in the realization of an efficient optomechanical rectifier/transistor.[63]

We now present the impact of the cavity decay rate on the OPO-modified setup in the third row of Figure 3c,f. It shows the robustness of the irreversibility associated with the OPO crystal non-linearity against the cavity decay rate κ. For a small number of thermal excitations ($n_b=10$), the impact of the decay rate remains minimal up to $|\chi|\approx\omega_b$. We observe a reduction in the entropy production rate as κ increases, and it grows linearly with respect to $|\chi|/\omega_b$. We remark that the entropy production rate diverges when $\kappa<\pm|\chi|\cos(\phi)$ due to the instability induced in the optomechanical system by the strength of the nonlinearity, i.e., $|\chi|\cos(\phi)$.

3.2. Quantum Correlations

Let us now proceed to analyze how the presence of an OPO in an optomechanical cavity influences the correlation profiles. For a bipartite quantum state ρ_{ab}, the net correlations between the

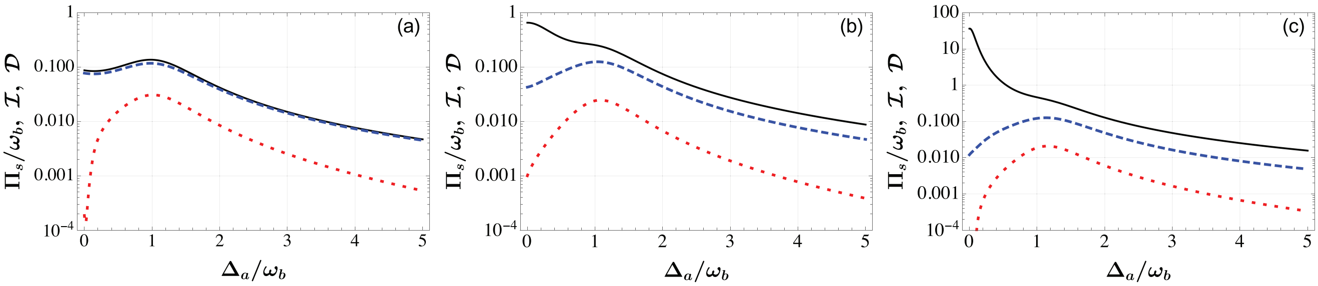

Figure 4. Entropy production rate Π_s (solid black curve), mutual information $\mathcal{I}$ (dashed blue curve) and quantum discord $\mathcal{D}$ (dotted red curve) as function of normalized detuning Δ_a/ω_b for different values of self nonlinearity a) $|\chi|=0$, b) $|\chi|=0.3\,\omega_b$, and c) $|\chi|=0.5\,\omega_b$. The other parameters are $\phi=0.8\,\pi$, $\gamma=10^{-2}\omega_b$, $G=0.1\,\omega_b$, $\kappa=0.5\,\omega_b$, and $n_b=10$.

two modes can be quantified by means of the quantum mutual information,

$$\mathcal{I}(\rho_{a:b}) = S(\rho_a) + S(\rho_b) - S(\rho_{ab}) \tag{13}$$

where $S(\rho_i)=-\mathrm{tr}\rho_i \ln \rho_i$ $(i=a,b)$ is the von Neumann entropy, and $\rho_a=\mathrm{tr}_b\rho_{ab}$ and $\rho_b=\mathrm{tr}_a\rho_{ab}$ are the reduced states of the joint two-mode state ρ_{ab}. However, considering the Gaussian nature of the states presented here, which are completely characterized by the two-mode covariance matrix, see Equation (7),

$$\mathcal{V}_{ab} = \begin{pmatrix} \mathcal{V}_a & c_{ab} \\ c_{ab}^T & \mathcal{V}_b \end{pmatrix} \tag{14}$$

where $\mathcal{V}_a$ ($\mathcal{V}_b$) is the covariance matrix of the optical cavity (mechanical) mode. The Gaussian distribution legitimise the use of Rényi-2 entropy, which is ref. [64]

$$S_2(\mathcal{V}_{ab}) = \frac{1}{2}\ln(\det\mathcal{V}_{ab}) \tag{15}$$

Thus, the Gaussian Rényi-2 mutual information for the two mode Gaussian state reads[64]

$$\mathcal{I}(\mathcal{V}_{a:b}) = \frac{1}{2}\ln\left(\frac{\det\mathcal{V}_a \det\mathcal{V}_b}{\det\mathcal{V}_{ab}}\right) \tag{16}$$

Next, we consider the measure of quantum discord based on the R'enyi-2 entropy, which quantifies the amount of quantum correlations beyond entanglement in a Gaussian state. Quantum discord is defined as the difference between the mutual information $\mathcal{I}(\mathcal{V}a:b)$ and the one-way classical correlations $\mathcal{J}(\mathcal{V}a|b)$.

$$\mathcal{D}(\mathcal{V}_{a|b}) = \mathcal{I}(\mathcal{V}_{a:b}) - \mathcal{J}(\mathcal{V}_{a|b}) \tag{17}$$

where $\mathcal{J}(\mathcal{V}a|b) = \sup\pi_b(X)S(\mathcal{V}a) - \int \mathrm{d}X p_X S(\mathcal{V}^{\pi_b}a|X)$ represents the maximum decrease in the R'enyi-2 entropy of subsystem a, when a Gaussian measurement has been performed on subsystem b such that $\pi_b(X) \geq 0$, and $\int \mathrm{d}X, \pi_b(X) = \mathbb{1}$. When considering the maximization over all possible measurements implemented on mode b, this can be expressed as

$$\mathcal{D}(\mathcal{V}_{a|b}) = \frac{1}{2}\ln(\det\mathcal{V}_b) - \frac{1}{2}\ln(\det\mathcal{V}_{ab}) + \inf_{\pi_b}\frac{1}{2}\ln(\det\mathcal{V}_a^{\pi_b}) \tag{18}$$

It has recently been demonstrated that the irreversibility generated by the steady state and the total amount of correlations shared between two coupled oscillators are closely related.[40] In what follows, we focus on the mutual information and quantum discord between the two modes at the stationary state as well as the influence of self nonlinear interaction on them. In **Figure 4**, we compare the entropy production rate π_S to the correlations established by the optomechanical system with a driven nonlinear crystal, as quantified by the mutual information $\mathcal{I}$ and quantum discord $\mathcal{D}$ at the phase $\phi=0.8\pi$. In Figure 4a, for $|\chi|=0$, we see a close similarity between the entropy production rate and the mutual information curves. For $|\chi|\neq 0$, Figure 4b,c, there is a striking difference between the entropy production rate and the quantum correlations. It can be seen that $0<\Delta_a/\omega_b<1$, the entropy production Π_s decreases while the mutual information $\mathcal{I}$ and discord $\mathcal{D}$ are increasing. This deviation from the case of $\chi=0$, see,[40] is due to the modification of the cavity decay rate by the presence of nonlinear medium. Therefore, we remark that the present study highlights the importance of understanding the versatile behavior of optomechanical system when optical parametric oscillator is placed inside the cavity. We find that nonlinear crystal in the optomechanical cavity modifies the irreversibility and quantum correlations. In contrast to the previous studies, see refs. [40, 44, 45], we have demonstrated that the optomechanical system entropy flow (increase or decrease), as well as its correlations can be manipulated by choosing the appropriate phase of the self-induced nonlinearity.

4. Conclusion

We have investigated the irreversible entropy generated in an interacting nonlinear hybrid quantum system by a stationary driven dissipation process. We studied the system of a nonlinear crystal placed inside a driven optomechanical cavity that can be described by two-mode composite Gaussian system. We showed that the stationary state entropy production rate depends on the strength of the nonlinear self interaction of the optical cavity mode. Our analysis demonstrated that the squeezed cavity mode induced by the presence of the nonlinear crystal modifies the entropy production rate and quantum correlations in an optomechanical system. We have further shown that the relationship between the entropy production rate and the quantum correlations is drastically modified by the nonlinear medium for small detuning. We remark that our investigation can easily be implemented in the current state-of-art experimental technology. Our work would benefit the current effort toward optimization of quantum thermal devices[12,23,65] and the better understanding energetic cost of cooling optomechanical systems.[66]

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

entropy production, optical parametric oscillators, optomechanical systems, quantum correlations

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