FUZZY SUMUDU TRANSFORM

by

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s	Sumudu transform
$\widetilde{f}$	Fuzzy function
s^{-1}	Inverse Sumudu transform
$\mathscr{L}$	Fuzzy Laplace transform
$\widetilde{u}$	Fuzzy number
$\mathbb{R}$	Real number
$\mathscr{F}(\mathbb{R})$	Fuzzy real number
α	Alpha level
$\widetilde{u}^{\alpha}$	Alpha level set of a fuzzy number
$\underline{u}^{\alpha}$	Lower bound of alpha level set of a fuzzy number
$\overline{u}^{\alpha}$	Upper bound of alpha level set of a fuzzy number
$-^{H}$	Hukuhara difference
$\mathbb{C}^{\mathscr{F}}$	Continuous fuzzy functions
$L^{\mathscr{F}}$	Lebesgue integrable fuzzy functions
$\mathbb{S}$	Fuzzy Sumudu transform
${}^{C}D^{\beta}$	Caputo fuzzy fractional derivative of order β
w_x	Fuzzy partial derivative of w with respect to x
w_t	Fuzzy partial derivative of w with respect to t
s_t	Sumudu transform with respect to variable t

LIST OF ABBREVIATIONS

CST	Classical Sumudu transform
FST	Fuzzy Sumudu transform
FLT	Fuzzy Laplace transform
ODE	Ordinary differential equation
FIVP	Fuzzy initial value problem
FDE	Fuzzy differential equation
FrDE	Fractional differential equation
FFDE	Fuzzy fractional differential equation
PDE	Partial differential equation
FPDE	Fuzzy partial differential equation
FCC	Fuzzy constant coefficient
SLDE	System of linear differential equation
SLFDE	System of linear fuzzy differential equation

Jelmaan Sumudu Kabur

ABSTRAK

Persamaan pembezaan biasa (PPB) telah lama digunakan untuk memodelkan masalah-masalah di dunia nyata. Dari masa ke masa, para penyelidik menyedari bahawa PPB bukan lagi kaedah pemodelan yang terbaik kerana ia kurang tepat, terutamanya ketika mengendalikan perkara-perkara yang dikelilingi ketidakpastian. Untuk berdepan dengan masalah ini, para penyelidik menggunakan teori set kabur untuk mencadangkan persamaan pembezaan kabur (PPK). Tidak seperti PPB yang gagal mengendalikan ketidakpastian pada nilai-nilai awal, PPK mampu mengendalikan masalah ini dengan cekap. Walau bagaimanapun, kaedah-kaedah yang boleh digunakan untuk mengendalikan PPK adalah sangat terhad, dan kebanyakannya masih di peringkat pembangunan. Dalam kajian ini, satu kaedah analitik yang baru akan dicadangkan untuk menyelesaikan suatu kelas PPK, termasuklah PPK berperingkat integer, persamaan pembezaan pecahan kabur (PPPK), persamaan pembezaan separa kabur (PPSK) dan sistem persamaan pembezaan kabur linear (SPPKL). Untuk tujuan ini, satu jelmaan kamiran yang baru iaitu jelmaan Sumudu kabur (JSK) dicadangkan. Hal ini dilakukan dengan mengintegrasikan teori set kabur dan jelmaan Sumudu klasik (JSKl). Sifat-sifat baru dan hasil-hasil asas berkenaan JSK akan dicadangkan dan dibuktikan secara matematik. Sifat-sifat baru ini termasuklah hasil berkenaan kedualan dengan jelmaan Laplace kabur (JLK), kelinearan, penskalaan, penganjakan, perlingkaran dan juga teorem-teorem berkenaan terbitan-terbitan kabur. Hasil-hasil yang dikemukakan kemudiannya digunakan untuk membangunkan prosedur-prosedur bagi menyelesaikan PPK, PPPK, PPSK dan SPPKL, yang mana kelas PPK ditafsirkan melalui konsep terbitan teritlak kuat. Dari situ, prosedur-prosedur yang dibangunkan didemonstrasikan ke atas beberapa contoh untuk menggambarkan kebolehgunaan JSK. Beberapa analisis telah dilakukan yang mana perilaku solusi-solusi yang ditemui digambarkan secara terperinci. Tambahan, perbandingan juga telah dilakukan antara solusi-solusi yang ditemui untuk kelas PPK dan solusi-solusi yang ditemui menggunakan PPB, merangkumi PPB berperingkat integer, persamaan pembezaan pecahan (PPP), persamaan pembezaan separa (PPS) dan sistem persamaan pembezaan linear (SPPL). Kelebihan utama JSK yang ditekankan di dalam tesis ini ialah ia memiliki sifat pengekalan skala. Ini bermakna fungsi yang telah dijelmakan, dengan domain baru, adalah menyerupai fungsi asal. Tambahan lagi, kajian ini menyediakan satu alternatif kepada para penyelidik ketika mengendalikan kelas PPK.

Fuzzy Sumudu Transform

ABSTRACT

Ordinary differential Equations (ODEs) have long been used to model real-life problems. As times goes on, researchers realized that ODEs are no longer the best modelling tool, since they are inaccurate, especially when dealing with stuff that are surrounded by uncertainties. To deal with this, researchers utilized the fuzzy set theory to propose fuzzy differential equations (FDEs). Unlike ODEs which could not cope with uncertainties at the initial values, FDEs handle this efficiently. However, the methods that can be used to deal with FDEs are very limited, and many of them are still at development stages. In this research, a new analytical method is proposed for solving a class of FDEs, including FDEs of integer order, fuzzy fractional differential equations (FFDEs), fuzzy partial differential equations (FPDEs) and system of linear fuzzy differential equations (SLFDEs). For this purpose, a novel integral transform called fuzzy Sumudu transform (FST) is proposed. This is done by integrating the fuzzy set theory and the classical Sumudu transform (CST). New properties and fundamental results concerning FST are proposed and proved mathematically. These new properties comprised results on duality with fuzzy Laplace transform (FLT), linearity, preserving, shifting, convolution, as well as theorems for fuzzy derivatives. The proposed results are then used to construct detailed procedures for solving FDEs, FFDEs, FPDEs and SLFDEs where the class of FDEs are interpreted under the strongly generalized differentiability concept. From there, the procedures are demonstrated on some examples, in order to illustrate FST applicability. Some analyses are performed where the behaviour of the solutions obtained are described in particulars. Furthermore, a comparison between the solutions obtained for the class of FDEs are compared with the solutions obtained using ODEs, where the ODEs include integer order ODEs, fractional differential equations (FrDEs), partial differential equations (PDEs) and system of linear differential equations (SLDEs). The main advantage of FST highlighted in this thesis is that it possessed the scale preserving property. This means that the transformed function, with a new domain, is a similitude to the original function. Additionally, this research provides an alternative for researchers when dealing with the class of FDEs.

CHAPTER 1

INTRODUCTION

1.1 Research Overview

On the subject of ordinary differential equations (ODEs), finding solutions using integral transforms is not a new topic. Integral transforms are one of the most crucial subject in operational calculus. They are mathematical operators that have been used widely to solve many problems in mathematics, physics and engineering (Davis, McNamara, Cottrell, & Campos, 2000; Namias, 1980; Saitoh, 1983). The precursor of the integral transforms is Fourier transform, which is used to express functions in finite interval. Subsequently, the concept was expanded to remove the necessity of finite intervals, and this led to a number of works on the theories and applications of integral transforms, some of which are Laplace (Bellman, Kalaba, & Lockett, 1966), Mellin (Tranter, 1948) and Hankel (Layman, 2001). Integral transforms have been widely used by researchers for solving models of ODEs. Not only on ODEs of integer order, but also on fractional differential equations (FrDEs), partial differential equations (PDEs) and system of linear differential equations (SLDEs). Integral transforms are useful as they allow an unknown function to be replaced by a suitable integral of this function that also contains a parameter. It can be expected that ODEs used in the problem are reduced to an algebraic expression that is much simpler for us to solve. The solution of the algebraic expression is then transformed back to its original parameter. This is very useful when dealing with complex problem as the domain after the transformation is simplified. It may seems a longer computation involved, but the advantage is that the solution is more straightforward.

Later, Watugala (1993) pioneered a new integral transform in the literature namely classical Sumudu transform (CST), and its application on ODEs in control engineering problem. Furthermore, Weerakoon (1994) extended the use of CST on PDEs, followed by his work on complex inversion formula (Weerakoon, 1998). The work is then continued by Asiru (2001), discussing on the convolution theorem of CST which can be expressed as polynomial and convergent infinite series. Later on, Belgacem, Karaballi, and Kalla (2002) emphasized on Laplace-Sumudu duality which is a vital step in building the theorems and properties of CST. At instance, the duality has been used to invoke a complex inverse of CST, as a Bromwich contour integral formula in the work of Belgacem and Karaballi (2005). The theories and applications of CST have been examined and explored by many authors (Agwa, Ali, & Kılıçman, 2012; Asiru, 2002; Eltayeb & Kılıçman, 2010; Kılıçman, Eltayeb, & Ismail, 2012). Some of recent works about CST are on the application of CST to solid and porous fin by Patel and Meher (2016) and on solving time-fractional Cauchy reaction-diffusion equation by Wang and Liu (2016). In the paper by Belgacem, Al-Shemas, and Silambarasan (2016), the authors used CST on the transient magnetic field in a lossy medium.

Despite few attentions given to CST, the integral transform performs well in its field. Comparing to other integral transforms, this integral transform has many properties which cause its visualization effortless. CST in general, is used in finding the solution of ODEs. Due to its unity property, the integral transform eases the process of finding the solutions. CST is more powerful compare to other integral transforms because the function transformed is similar to the resulting function.

As CST can act as a potent tool for ODEs, it provides a useful technique in solving many real-life systems. ODEs have long been used to model real physical phenomenon. However, it can never be confirmed whether the model is perfect or not. For example, the initial value of an ODE might not be known accurately. The initial value might take any value with uncertainty. For instance, the initial value x_0 might be "less than x_0", "about x_0" or "more than x_0". If this is the case, ODE has failed to cope with this situation. Thus, other theories are needed to tackle the situation. One of the popular theories for describing such situation is the fuzzy set theory proposed by Zadeh (1965). Later, the development of the fuzzy set led to fuzzy derivative concept, introduced by Chang and Zadeh (1972), focusing on fuzzy mapping with additional properties; derivatives and integrals. A vast literature of work on fuzzy set and fuzzy derivatives have been proposed and introduced in the literature (Dubois & Prade, 1982; Goetschel & Voxman, 1986; Kaleva, 1987). For example, Puri and Ralescu (1983) used Rådström embedding theorem to define the concept of differentials of a fuzzy function. While Ding, Ma, and Kandel (1997) studied the solutions of fuzzy differential equations (FDEs) with parameters by the topological degree method. In the paper written by Agarwal, Lakshmikantham, and Nieto (2010), the authors introduced fuzzy fractional differential equations (FFDEs). They proved the existences of the solutions with and without the Hausdorff measure of non-compactness. Several recent studies involved papers on FDEs in the quotient space of fuzzy numbers by Qiu, Zhang, and Lu (2016) and the numerical solutions of FDEs using variational iteration method by Hosseini, Saberirad, and Davvaz (2016). From there, studies on model with several FDEs simultaneously are also done, where it is referred to as system of linear fuzzy differential equations (SLFDEs). This can be seen in several articles in the literature (Fard & Ghal-Eh, 2011; Najariyan & Mazandarani, 2015).

Recently in this field, Allahviranloo and Ahmadi (2010) proposed a fuzzification of the Laplace transform, termed fuzzy Laplace transform (FLT). The FLT is proven to be capable in solving FDEs effectively. In solving fuzzy partial differential equations (FPDEs), Salahshour and Haghi (2010) used FLT to handle fuzzy heat equations, while ElJaoui and Melliani (2016) provided a study on a more general version of FPDEs. In the application of FLT on FDEs of integer order, the FLT is used to solve second order linear FDEs and the state-space description of fuzzy linear continuous-time systems by Salahshour and Allahviranloo (2013) before then, Muhammad Ali and Haydar (2013) extended the work to FDEs of third order. Meanwhile, Salahshour, Khezerloo, Hajighasemi, and Khorasany (2012) applied FLT for solving fuzzy integral equations. Despite the popularity of FLT, the inverses of FLT, as well as Laplace transform often involve complex calculations. This can be seen as a disadvantage, since latest integral transform like CST is able to reduce the complexity during calculation. In addition, the scale preserving property of CST makes it more powerful then Laplace transform. So, the effort to introduce CST into fuzzy settings is of a great interest.

1.2 Problem Identification

FDEs have been proven to be an effective way to model real-world problems. This is becoming handy, especially when dealing with complex problems where it is hard to model them using ODEs or stochastic differential equations. Using FDEs, uncertainties that surround the phenomena can be integrated into the model itself. Specifically, FDEs take account the uncertainties that surround the initial values in models. Since the introduction of FDEs, researchers also started to develop solving methods. In order to propose the solving methods, researchers proposed several interpretation of FDEs. This includes the interpretation under the strongly generalized differentiability concept that is used through out this thesis. However, in the literature, there are still very limited methods that can be used when considering this type of interpretation for FDEs. From this, several methods have been introduced for solving FDEs. For examples, Ma, Friedman, and Kandel (1999) developed an algorithm based on classical Euler method to solve FDEs and Babolian, Sadeghi, and Javadi (2004) utilized Adomian method instead. While in the paper by Allahviranloo (2004), Runge-Kutta method has been used to solve FDEs. Recently, Allahviranloo and Ahmadi (2010) have proposed an analytical method, called FLT to solve FDEs. This is done by integrating Laplace transform into fuzzy setting. Since FLT is an integral transform, the solutions are more straightforward and do not involve approximations. Even so, the methods proposed are still at the development stage.

There are several more integral transformation methods exist in the literature. One of the most recent is the CST proposed by Watugala (1993, 1998). The CST has been proven to have some advantages over other integral transforms. For example, it has a unity property that helps making the transformation process easier. By utilizing this property, some times, the unit of the final solution can be obtained by halfway solving it. This happen when applying CST to ODEs. However, CST cannot be used directly to solve FDEs since it does not adapt fuzzy settings. Therefore, it becomes necessary for researchers to study CST in the fuzzy environment.

From this discussion, it can be concluded that there are some issues that need to be focused. The issues are as follows.

(i) Whenever FDEs are interpreted using strongly generalized differentiability concept, solving methods are limited.

(ii) The existing methods in the literature for FDEs interpreted using strongly generalized differentiability concept are still at development stages and further works need to be conducted.

(iii) Specifically, analytical methods for FDEs under this interpretation are very limited. Recent method is FLT, which is an extension of Laplace transform into fuzzy settings.

(iv) Other analytical methods in solving ODEs have potential to be studied in fuzzy settings. This includes a recent integral transform in the literature, CST.

1.3 Research Objectives

The objectives of this research are as follows.

(i) To propose a new integral transform in fuzzy setting, called fuzzy Sumudu transform (FST), and its fundamental results.

(ii) To construct detailed procedures for solving FDEs of integer order, FFDEs, FPDEs and SLFDEs using FST.

(iii) To analyze the results obtained for the class of FDEs considered based on the observation on graphs and tables.

(iv) To compare the results obtained using FST on the class of FDEs with the results of the class of ODEs, comprising ODEs of integer order, FrDEs, PDEs and SLDEs.

1.4 Scope and Limitation of Research

This research focuses to extend the applicability of CST in fuzzy settings. For this purpose, fuzzification of CST is done which takes account Zadeh's fuzzy set theory. The fuzzified version of CST is then termed FST. Some fundamental results of FST are also proposed and proved. From there, the solving procedures are developed in details. Applications on the class of FDEs, including FDEs of integer order, FFDEs, FPDEs and SLFDEs are also done afterwards. The class of FDEs considered in this thesis is interpreted under the strongly generalized differentiability concept. Since the method proposed is by utilizing an integral transform, it is only used for solving linear problems.

1.5 Thesis Outline

The subsequent chapters in this thesis are organized as follow. In Chapter 2, some theoretical backgrounds on fuzzy number, fuzzy function and fuzzy derivative are presented. This would help to understand the rest of this thesis. Later in Chapter 3, FST is introduced and some properties and theorems are proposed alongside. These include results on duality with FLT, linearity, scale preservation, shifting and convolution theorems. Additionally, several theorems for first degree derivatives, fractional derivatives, and partial derivatives are also introduced in this chapter. Later in Chapter 4, procedures for solving FDEs of integer order are provided, together with demonstration on two examples. While in Chapter 5, this thesis provides the procedures for solving FFDEs. They are then demonstrated on two examples. In Chapter 6, procedures for finding the solution of FPDEs are constructed, where they are applied on two examples afterwards. Next, Chapter 7 provides the procedures for handling SLFDEs. Two examples is also given to show the practicability of the method. All the results obtained in Chapter 4, 5, 6, 7 are analyzed in details in corresponding chapters. Comparisons between results obtained under the class of FDEs and ODEs are also performed. Finally, in Chapter 8, conclusion is drawn and the research contributions are highlighted. Some potential research directions are stated as well. This could be a guide for researchers to further extend the exploration of this field.

1.6 Summary

In this chapter, an overview of this research has been provided. From there, some backgrounds and motivations of this research are given. Next, the problems are identified and highlighted. The significance of this research has been shown as well. Then, objectives of this research are developed. There are four objectives considered, which include results on definition and fundamental properties of FST, applications of FST to the class of FDEs, analysis of the results obtained using FST and comparison between results obtained using FST and results of the class of ODEs. Scope and limitation of this research are then stressed. The limitation stated opened up potential research direction in the future. Finally, the outline of this thesis is provided in details.