NUMERICAL SIMULATION OF THE BURGERS' EQUATION

by

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LIST OF ABBREVIATIONS

PDE	Partial differential equation
E-EFDM	Explicit exponential finite difference method
I-EFDM	Implicit exponential finite difference method

Simulasi Berangka Untuk Persamaan Burgers’

ABSTRAK

Dalam kajian ini, kaedah ekponen beza terhingga telah digunakan untuk menyelesaikan persamaan Burgers’ dalam satu dimensi dengan nilai h (saiz langkah) yang berbeza. Persamaan Burgers’ digunakan dalam kajian ini kerana persamaan dapat mengawal mudah proses resapan tidak linear. Persamaan Burgers’ adalah tidak linear transformasi Hopf-Cole digunakan untuk menukar persamaan haba satu dimensi. Oleh itu, kaedah eksponen beza terhingga digunakan untuk menyelesaikan persamaan haba satu dimensi. Tiga teknik tersebut ialah kaedah tersirat (explicit) ekponen beza terhingga, kaedah tak tersirat (implicit) ekponen beza terhingga dan persamaan Burgers’ yang diubah menggunakan kaedah ekponen beza terhingga. Dalam proses penyelesaian, kaedah tersirat exponen beza terhingga digunakan secara langsung untuk menyelesaikan persamaan Burgers’ manakala kaedah tak tersirat eksponen beza terhingga membawa kepada sistem persamaan linear. Pada setiap peringkat masa, kaedah Newton digunakan untuk menyelesaikan persamaan linear ini. Dalam penyelesaian satu dimensi persamaan Burgers’ diubah mengunakan kaedah ekponen beza terhingga. Proses penyelesaiannya ialah discretize pada terbitan masa dan terbitan ruang menggunakan kaedah ekponen beza terhingga. Keputusan penyelesaian berangka bagi setiap kaedah dibandingkan dengan penyelesaian tepat dan penyelesaian yang diperolehi menggunakan ketiga-tiga kaedah ini adalah tepat dan boleh dipercayai. Peratus kesilapan-kesilapan adalah dikira dan didapati cukup kecil.

Numerical Simulation of the Burgers' Equation

ABSTRACT

In this study, the exponential finite difference technique has been used to solve one-dimensional Burgers' equation with different value of *h* (step size). Burgers' equation is considered in this study because the equation governing simple nonlinear diffusion process. Since Burgers' equation is nonlinear equation, the Hopf-Cole transformation has been applied to convert the equation to one-dimensional heat equation. Consequently, the exponential finite difference method has been used to generate of one-dimensional heat equation. Three techniques called explicit exponential finite difference method, implicit exponential finite difference method and modified Burgers' equation using explicit exponential finite difference method have been implemented. In the solution process, the explicit exponential finite difference method used a direct to solve the Burgers' equation while the implicit exponential finite difference method leads to a system of nonlinear equation. At each time-level, Newton's method is used to solve the nonlinear system. The solution of the one-dimensional modified Burgers' equation by using the explicit exponential finite difference method. The solution process have been discretized the time derivative and spatial derivative using exponential finite difference technique. Numerical solutions for each method are compared with exact solution and the results obtained using three methods are precise and reliable. The percent errors are compute and found to be sufficiently small.

CHAPTER 1

INTRODUCTION

1.1 Overview

Partial differential equations (PDEs) are considerably more demanding, and can challenge the analytical skills of even the most accomplished mathematician. Many of the most effective solution strategies rely on reducing the PDE to one or more ordinary differential equations. Thus, in the course of our study on PDEs, we will need to develop some of the more advanced aspects of the theory of ordinary differential equations, including boundary value problems, eigenvalue problems, series solutions, singular points, and special functions (Olver, 2014). PDEs also have an important application in fields of engineering and physics such as mechanical engineering and chemical engineering (Handschuh, 1987). PDEs are derived into characteristics. There are parabolic equation, elliptic equation and hyperbolic equation. The classifications of PDEs are depending on the coefficients of the highest order derivative appears in PDEs. In this dissertation, Burgers' equation is considered as PDEs to be solved numerically. It is because Burgers' equation governing simple nonlinear diffusion process in PDEs (Hoffmann and Chiang, 2000).

The Burgers' equation was initially given by Bateman in 1915 (Wani and Thakar, 2013) and later rediscovered by Burgers' as a model of turbulence in 1939 (Inan and Bahadir, 2013b). Burgers' equation is an important and simple model for the understanding of physical flows. Burgers' equation is found to describe various kinds of phenomena such as mathematical model of turbulence and the approximate theory of flow through a shock wave travelling in a viscous fluid (Inan and Bahadir, 2013a).

In the dynamic field of application, Hopf in 1950 and Cole in 1951 reported independently that the Burgers' equation can be transformed to linear diffusion equation and it can be solved exactly for arbitrary initial conditions (Dag, Irk and Saka, 2005). The Burgers' equation can be solved exactly by a restricted set of initial function only because it is second order nonlinear partial differential equations. This equation has been applied to mass transportation, the formation order of a shock wave, boundary layer behavior and simple model of turbulence (Saka and Dag, 2008).

Burgers' equations provide the simplest nonlinear models of turbulence in the phenomena process. The reaction diffusion and convection diffusion systems have important features namely the existence of a delay time and relaxation time (Jawad, Petkovit and Biwas, 2010). The solution of Burgers' equation series converges very slow for small values of viscosity (Dhawan, 2011).

In the study of Burgers' equation, many numerical methods have been proposed and implemented to obtain the approximate solution of this equation. Numerical techniques based on finite difference method (Kutluay, Bahadir and Odzes, 1999; Gulsu and Ozis, 2005; Kadalbajoo and Awashi, 2006; Gulsu, 2006), finite element method (Ali, Gardner and Gardner, 1992; Kutluay, Esen and Dag, 2004; Mittal and Jain, 2012; Soliman, 2012), and boundary element method (Bahadir and Saglam, 2005) have been developed in effort to solve Burgers' equation numerically.

In this study, two types of finite difference method will be used to solve Burgers' equation. There are explicit exponential finite difference method (E-EFDM) and implicit exponential finite difference method (I-EFDM). Besides that, modified

Burgers' equation also will be solved by using E-EFDM and I-EFDM. In an addition, modified E-EFDM will be used to solve the modified equation.

1.2 Problem Statement

Burgers' equation described numbers of phenomena such as a mathematical model of turbulence and the approximate theory of flow through a shock wave traveling in a fluid. Among available numerical techniques for solving Burgers' equation, two types of finite difference methods has been considered in this study, i.e. explicit exponential finite difference method (E-EFDM) and implicit exponential finite difference method (I-EFDM). Besides that, E-EFDM will be applied to solve modified Burgers' equation. There is no current literature on investigating the accuracy of these three techniques. Based on this fact, this study is therefore conducted.

1.3 Objectives

The objectives of this study involved three main stages:

a) To discretized the Burgers' equation using E-EFDM and I-EFDM and to descretize modified Burgers' equation using E-EFDM.
b) To verify the accuracy of the numerical solutions of Burgers' equation and modified Burgers' equation generated by E-EFDM and I-EFDM.
c) To compare the accuracy of the numerical solution generated by E-EFDM and I-EFDM.

1.4 Scope of Study

In this study, the numerical solutions of one-dimensional Burgers' equation are generated by E-EFDM and I-EFDM. Meanwhile, modified Burgers' equation is solved by using E-EFDM. Due to complexity of the equation, the stability of the exponential finite difference methods will not be discussed in this study.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter, the methods have been used to solve Burgers' equation and modified Burgers' equation is reviewed. The linearization of the Burgers' equation is also discussed. Consequently, the exact solution of the linearize equation is considered in the following section. Finally, the discussions about numerical methods are summarized.

2.2 Burgers' Equation

Burgers' equation began in 1915 when Bateman introduced this equation in the physical context (Bahadir, 1999). The series solution is one of the most interesting solutions of Burgers' equation when Fay obtained it from an acoustic framework in 1931. In 1940, Burgers' equation emphasized its importance and provides a special solution to it. After 8 years, Burgers' concluded its form as a model in the theory of turbulence. Lagerström et al. (1949) found that the Burgers' equation can be transformed to the linear heat equation (Cole, 1951). Hopf in 1950 independently discovered this transformation where it is known as the Cole-Hopf transformation. Fay's series solution was rediscovered by Cole (1951) as an approximate solution of the Burgers' equation with sinusoidal initial condition. After that, Lighthill (1956) and Blackstock (1964) used Burgers' equation to study the propagation of one-dimensional acoustic signal with finite amplitude. Hayes (1958) used the Burgers' equation to discuss the shock structure Navier-Stokes fluid. Benton (1966) and Benton (1967) found an exact solution of the Burgers' equation. In 1970, Rodin found a solution to the Riccati Burgers' equation without using any additional conditions (Abd-el-Malek & El-Mansi, 2000).

Burgers' equation was solved analytically for the values of limited initial conditions (Gao, Le & Shi, 2013). Benton and Platzman published 35 different solutions to the initial value problem of Burgers' equation in the infinite domain as well as two other solutions for the initial and boundary value problems in the finite domain (Mukundan & Awasthi, 2015). Morgan-Michal methods can be applied to determine the proper groups for Burgers' equation, without taking into the account the additional requirements (Abd-el-Malek & El-Mansi, 2000).

The one-dimensional non-linear Burgers' equation is given as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-v\frac{\partial^2 u}{\partial x^2}=0, \qquad d<x<e, \tag{2.1}$$

with initial condition

$$u(x,0)=g(x)$$

and boundary conditions

$$u(d,t)=h_1(t)$$

$$u(e,t)=h_2(t),$$

for $t>0$, where v is the positive coefficient of kinematic viscosity and g, h_1 and h_2 are variable for the prescribed functions (Inan & Bahadir, 2013a).

2.3 Linearization and Exact Solution of the Burgers' Equation

In this section, the exact solution of the Burgers' equation is generated by reducing the equation to linear heat equation using Hopf-Cole transformation. The Hopf-Cole transformation is defined by the following theorem:

Theorem 1 (Inan & Bahadir, 2013b)

If $\theta(x,t)$ is any solution of the one-dimensional linear heat equation,

$$\frac{\partial \theta}{\partial t} = v\frac{\partial^2 \theta}{\partial x^2} \tag{2.2}$$

Then, the following Hopf-Cole transformation is assumed as a solution to equation (2.1).

$$u(x,t) = -2v\frac{\theta_x}{\theta} \tag{2.3}$$

Therefore, Theorem 1 is considered where the solution of Burgers' equation can be obtained by solving equation (2.2) with initial condition,

$$\theta(x,0) = \exp\left(-\frac{1}{2v}\int_0^x u_0(\xi)d\xi\right),\ 0 < x < 1 \tag{2.4}$$

and boundary condition

$$\theta_x(0,t) = \theta_x(1,t) = 0\,. \tag{2.5}$$

The Fourier series solution to Equation (2.5) is given as (Inan & Bahadir, 2013b)

$$\theta(x,t) = A_0 + \sum_{n=1}^{\infty}\exp\left(-vn^2\pi^2 t\right)A_n\cos(n\pi x) \tag{2.6}$$

where

$$A_0 = \int_0^1 \exp\left\{-\frac{1}{2v}\int_0^x u_0(\xi)d\xi\right\}dx$$
$$A_n = 2\int_0^1 \exp\left\{-\frac{1}{2v}\int_0^x u_0(\xi)d\xi\right\}\cos(n\pi x)dx, n = 1,2,3,...$$

In order to apply Hopf-Cole transformation, equation (2.6) is differentiated with respect to x as

$$\theta_x(x,t) = -\pi\sum_{n=1}^{\infty} A_n n\exp(-vn^2\pi^2 t)\sin(n\pi x) \tag{2.7}$$

Consequently, solution (2.6) and its derivative, equation (2.7) are substituted into transformation (2.3) gives the following equation

$$u(x,t) = \frac{2\pi v\sum_{n=1}^{\infty} A_n n\exp(-vn^2\pi^2 t)\sin(n\pi x)}{A_0 + \sum_{n=1}^{\infty} A_n \exp(-vn^2\pi^2 t)\cos(n\pi x)} \tag{2.8}$$

where it is considered as the exact solution of Burgers' equation.

2.4 Modified Burgers' Equation

The general form of modified Burgers' equation (MBE) is given as

$$U_t + U^p U_x - vU_{xx} = 0, \qquad p = 1,2 \tag{2.9}$$

where U is dependent variable, t and x are independent variable, and v is a constant and $p = 1,2$. When $p = 1$, equation (2.9) is called Burgers' equation while when $p \geq 2$, the equation (2.9) is called modified Burgers' equation (Saka & Dag, 2008). The modified Burgers' equation is called the nonlinear advection-diffusion equation

Ramadan and El-Danaf (2005) studied the solution of the modified Burgers' equation by using the collocation method with quintic splines. A numerical solution for the convection-diffusion equation using El-Gendi method with interface points and then shown the numerical result for Burgers' and modified Burgers' equation was proposed by Temsah (2009). In 2009, Bratsos (2009) presented a finite difference scheme based on rational approximation to the matrix-exponential term in a two-time level recurrence relation as numerical method for solving modified Burgers' equation. A finite difference scheme based on fourth-order rational approximations to the matrix-exponential term in a two time level recurrence relation is developed by Bratsos (2011) to solve modified Burgers' equation. Roshan and Bhamra (2011) used a cubic B-spline function as the test function and a linear hat function as a trial function to solve modified Burgers' equation numerically by Petrov-Galerkin. Kutluay, Ucar and Yagmurlu (2015) applied the cubic B-spline collocation finite element method to develop a numerical technique for solving the modified Burgers' equation.

2.5 Numerical Methods

A study of Burgers' equation is important since it arises in the approximate theory of flow through a shock wave propagating in a viscous fluid and in the modeling of turbulence. Burgers' equation and the Navier-Stokes equation are similar in the form of their nonlinear terms and in the occurrence of higher order derivatives with small coefficients in both. The exact solutions of the one-dimensional Burgers' equation have been surveyed by Benton and Platzman (Soliman, 2012).

In order to approximate the solution of Burgers' equation, many numerical methods have been proposed and implemented. Burgers' equation can be treated as a qualitatively correct approximation of the Navier-Stocks equation. That means it can be considered as a simplified form of the Navier-Stokes equation. The characteristic feature of Burgers' equation is the combination of two terms $\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2}$ (diffusion term) and $u\frac{\partial u}{\partial x}$ (convection term), which give rise the appearance of dissipation layers (Kadalbajoo & Awasthi, 2006). The nonlinear term $u\frac{\partial u}{\partial x}$ makes it more interesting to learn which attracted many researchers to develop the solutions for Burgers' equation. Two different analytical solutions of Burgers' equation have been found for a restricted set of arbitrary initial and boundary conditions (Dhawan, 2011).

Rubin and Grave (1974) have used the cubic spline function and quasilinearization for the numerical solution of Burgers' equation. Abbasbandy and Darvishi (2005a, b) discussed the numerical solution of Burgers' equation using the Adomain methods. Jain and Holla (1976) used implicit finite difference schemes with splitting technique and interpolation cubic spline to obtain the numerical solution of Burgers' equation. Darvishi and Javidi (2006) studied the numerical solutions of the Burgers' equation using pseudospectral methods and Darvish's preconditioning. Numerical technique based on B-spline function was presented by Kapoor and Dhawan (2010). Dhawan (2011) used the basic functions of B-spline in solving one-dimensional Burgers' equation.

Many studies have been done to deal with numerical solutions of Burgers' equation for small value ν. Zhang, Wei and Kouri (1997) studied the Burgers' equation with high Reynolds number because it can produce very high accuracy while using small number of grid points. Gorguis (2006) solved the Burgers' equation by presenting a comparison between Cole-Hopf transformation and decomposition.

Kadalbajoo and Awasthi (2006) developed a stable numerical method based on the Crank-Nicolson scheme to solve Burgers' equation. Biazar and Aminikhah (2009) used iterative methods to solve variational Burgers' equation where the solution is found better than ADM which has been reported by Basto, Semiao and Calheiros

(2007). The Douglas difference scheme for solving Burgers' equation was investigated by Pandey, Verma & Verma (2009).

Caldwell, Wanless and Cook (1981) solved the Burgers' equation by changing the size of the elements on each level using information from three steps via the finite element method. Caldwell and Smith (1982) extended the work of Caldwell et al. (1981) to general case of element.

Abdou and Soliman (2005) solved one-dimensional Burgers' equation and coupled Burgers' equations using variational iterative methods. Wei and Gu (2002) applied a conjugate filter approach to solve the Burgers' equation. Dehghan, Asgar and Muhammad (2007) obtained the numerical results of coupled viscous Burgers' equation by using Adomain-Pade techniques. Zang and Wang (2012) solved the Burgers' equation using a predictor-corrector compact finite difference scheme.

The explicit exponential finite difference method for solving of heat equation was originally developed by Bhattacharya (1985). Bhattacharya (1990) and Handschuh and Keith (1992) solved Burgers' equation using the exponential finite difference method without using the Hopf-Cole transformation (Inan & Bahadir, 2015). Bahadir (2005) used the exponential finite difference techniques to solve Korteweg-de Vries (KdV) equation. Recently, the Crank-Nicolson exponential finite difference method was applied to solve Burgers' equation (Inan & Bahadir 2014).

In the following chapter, Burgers' equation will be solved numerically by explicit exponential finite difference method (E-EFDM) and the implicit exponential finite difference method (I-EFDM). Then, the E-EFDM will be performed for solving modified Burgers' equation.

CHAPTER 3

EXPLICIT EXPONENTIAL FINITE DIFFERENCE METHOD

3.1 Introduction

In Chapter 2, the exact solution of Burgers' equation is obtained using the Hopf-Cole transformation applied to the solution of one-dimensional linear heat equation. However, in this chapter, Burgers' equation is solved numerically by E-EFDM where the Hopf-Cole transformation for the solution of heat equation is still considered. Initially, the heat equation will be discretized using E-EFDM. Then, Hopf-Cole transformation will be used to generate the solution of Burgers' equation. In order to verify the proposed method, two problems will be tested.

3.2 Method of Solution

According to E-EFDM, the time and space derivation are discretized by using forward difference and central difference approximation. One-dimensional heat equation is considered as follows:

$$\frac{\partial \varphi}{\partial t} = v\frac{\partial^2 \varphi}{\partial x^2} \tag{3.1}$$

The solution of heat equation is assumed as the following product solution

$$\varphi(x,t) = \phi(x)\theta(t) \tag{3.2}$$

where

$$\frac{\partial \varphi}{\partial t} = \phi \frac{\partial \theta}{\partial t}, \tag{3.3a}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \theta \frac{\partial^2 \phi}{\partial x^2} \tag{3.3b}$$

Equation (3.3a) and (3.3b) then substitute into Equation (3.1) gives

$$\phi \frac{\partial \theta}{\partial t} = v\theta \frac{\partial^2 \phi}{\partial x^2} \tag{3.4}$$

Equation (3.4) involves two variables which can be separated as the following separable equation:

$$\frac{1}{\theta(t)} \frac{d\theta}{dt} = -\kappa \tag{3.5}$$

$$v \frac{1}{\phi(x)} \frac{d^2\phi}{dx^2} = -\kappa \tag{3.6}$$

where κ is a constant. Equation (3.5) and (3.6) are solved separately. In order to solve equation (3.5), the equation is multiplied by $\phi(x)$. The generated equation is rearranged and equation (3.2) and (3.3a) are considered. Thus, the following equation is obtained

$$\frac{1}{\varphi(x,t)} \frac{\partial \varphi}{\partial t} = -\kappa \tag{3.7}$$

At $t = 0$, this equation has a solution for non-zero value of $\varphi(x,t)$ which can be written as

$$\varphi(x,t) = \varphi(x,0)\exp(-\kappa t) \tag{3.8}$$

Next, equation (3.6) is going to be solved. Initially, equation (3.6) is multiplied by $\theta(t)$. By substituting equation (3.2) and (3.3b) into the generated equation, it gives

$$\frac{v}{\varphi(x,t)} \frac{\partial^2 \phi}{\partial x^2} = -\kappa \tag{3.9}$$